5 Triple integral

5.1 Definition and properties of triple integral

We used a double integral to integrate over a two-dimensional region and so it's natural that we'll use a triple integral to integrate over a three-dimensional region. Suppose the function of three variables $f(x, y, z)$ is defined in the three dimensional region V. Choose the whatever partition of the region V into n subregions

$$\Delta v_1,\ \Delta v_2,\ \ldots,\ \Delta v_k,\ \ldots, \Delta v_n$$

where Δv_k denotes the kth subregion, as well the volume of this subregion.

For each subregion, we pick a random point $P_k(\xi_k, \eta_k, \zeta_k) \in \Delta v_k$ to represent that subregion and find the product of the value of the function at that point and the volume of subregion $f(P_k)\Delta v_k$. Adding all these products, we obtain the sum

$$\sum_{k=0}^{n} f(P_k)\Delta v_k, \tag{5.1}$$

which is called the *integral sum* of the function $f(x, y, z)$ over the region V.

Let

$$\operatorname{diam} \Delta v_k = \max_{P,Q \in \Delta v_k} |\overrightarrow{PQ}|$$

be the diameter of the subregion Δv_k and λ the greatest diameter of the subregions, i.e.

$$\lambda = \max_{0 \le k \le n} \operatorname{diam} \Delta v_k$$

Definition. If there exists the limit

$$\lim_{\lambda \to 0} \sum_{k=0}^{n} f(P_k)\Delta v_k$$

and this limit doesn't depend on the partition of the region V and the choice of the points P_k in the subregions, then this limit is called the triple integral of the function $f(x, y, z)$ over the region V and denoted

$$\iiint\limits_V f(x, y, z)dxdydz$$

Thus, by this definition

$$\iiint\limits_V f(x, y, z)dxdydz = \lim_{\lambda \to 0} \sum_{k=0}^{n} f(P_k)\Delta v_k \tag{5.2}$$

If the function $f(x, y, z)$ is continuous in the closed region V, then the triple integral (5.2) always exists.

The properties of the triple integral are quite similar to the properties of the double integral.

Property 1.

$$\iiint\limits_V [f(x, y, z) \pm g(x, y, z)]dxdydz = \iiint\limits_V f(x, y, z)dxdydz \pm \iiint\limits_V g(x, y, z)dxdydz$$

Property 2. If c is a constant then

$$\iiint\limits_V cf(x, y, z)dxdydz = c\iiint\limits_V f(x, y, z)dxdydz$$

Property 3. If $V = V_1 \cup V_2$ and the regions V_1 and V_2 have no common interior point then

$$\iiint\limits_V f(x, y, z)dxdydz = \iiint\limits_{V_1} f(x, y, z)dxdydz + \iiint\limits_{V_2} f(x, y, z)dxdydz$$

Let $f(x, y, z) \geq 0$ be the density of a three-dimensional solid V at the point (x, y, z) inside the solid. By picking a point P_k to represent the subregion Δv_k we treat the density $f(P_k)$ constant in the subregion Δv_k and the product $f(P_k)\Delta v_k$ is the approximate mass of the subregion Δv_k. The approximate mass because we have substituted the variable density $f(x, y, z)$ by the constant density $f(P_k)$.

The integral sum is the sum of the approximate masses of the subregions, i.e. the approximate mass of the region V. The limiting process $\lambda \to 0$ means that all diameters of the subregions are infinitesimals. The density at the point P_k represents the density of the subregion Δv_k with the greater accuracy and the integral sum will approach to the total mass of the region V.

Therefore, if the function $f(x, y, z) \geq 0$ is the density of a three-dimensional solid V then the triple integral equals to the mass of the solid V

$$m = \iiint\limits_V f(x, y, z)dxdydz$$

If the region V has the uniform density 1, then the mass and volume are numerically equal, i.e. if $f(x, y, z) \equiv 1$, then the volume of the region V is computable by the formula

$$V = \iiint\limits_V dxdydz \qquad (5.3)$$

An example how to use this formula we have later.

5.2 Evaluation of triple integral

The region V in the space is called *regular in direction of z axis* if there are satisfied three conditions.

1. Any line parallel to the z axis passing the interior point of this region cuts the boundary surface at two points.
2. The projection of the region onto xy plane is a regular plain region.
3. Cutting the region by the plane parallel to some coordinate plane creates two regions satisfying the conditions 1. and 2.

If those conditions are fulfilled, then the region V is determined by inequalities $a \leq x \leq b$, $\varphi_1(x) \leq y \leq \varphi_2(x)$ and $\psi_1(x,y) \leq z \leq \psi_2(x,y)$. We can define the iterated integral

$$\int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} dy \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz$$

To compute this iterated integral we have to compute three definite integrals. First we integrate with respect to the variable z holding x and y constant

$$\Psi(x,y) = \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz$$

We call this inside integral and the result is a function of two variables $\Psi(x,y)$. Next we integrate with respect to the intermediate variable y holding x constant

$$\Phi(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} \Psi(x,y)dy$$

The result is a function of the one variable $\Phi(x)$. Finally we compute the outside integral

$$\int_a^b \Phi(x)dx$$

Notice that the limits of the outside variable a and b are always constants. The limits of the intermediate variable $\varphi_1(x)$ and $\varphi_2(x)$ depend in general on the outside variable. The limits of the inside variable $\psi_1(x,y)$ and $\psi_2(x,y)$ depend in general on the outside variable and on the intermediate variable.

Example 1. Compute the iterated integral $\int\limits_0^1 dx \int\limits_0^x dy \int\limits_0^{xy} (x+y)dz$

First we integrate with respect to the inner variable z. Since $x+y$ is constant, then

$$\int\limits_0^{xy} (x+y)dz = (x+y)\cdot z\Big|_0^{xy} = x^2y + xy^2$$

This result we integrate with respect to intermediate variable y

$$\int\limits_0^{x} (x^2y+xy^2)dy = x^2\cdot\frac{y^2}{2}\Big|_0^x + x\cdot\frac{y^3}{3}\Big|_0^x = \frac{x^4}{2}+\frac{x^4}{3} = \frac{5x^4}{6}$$

and finally with respect to x

$$\int\limits_0^1 \frac{5x^4}{6}dx = \frac{5}{6}\cdot\frac{x^5}{5}\Big|_0^1 = \frac{1}{6}$$

Since we have assumed that the projection of the region V onto xy plane is a regular plane region, then the region V can be determined by inequalities $c \le y \le d$, $\varphi_1(y) \le x \le \varphi_2(y)$ and $\psi_1(x,y) \le z \le \psi_2(x,y)$ and the iterated integral can be defined as

$$\int\limits_c^d dy \int\limits_{\varphi_1(y)}^{\varphi_2(y)} dx \int\limits_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz$$

Just like we have defined the regular region in direction of z axis, we can define the regular region in direction of x axis and the regular region in direction of y axis. In the first case it is possible to define the iterated integrals

$$\int\limits_a^b dy \int\limits_{\varphi_1(y)}^{\varphi_2(y)} dz \int\limits_{\psi_1(y,z)}^{\psi_2(y,z)} f(x,y,z)dx$$

or

$$\int_a^b dz \int_{\varphi_1(z)}^{\varphi_2(z)} dy \int_{\psi_1(x,y)}^{\psi_2(y,z)} f(x,y,z)dx$$

and in the second case

$$\int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} dz \int_{\psi_1(x,z)}^{\psi_2(x,z)} f(x,y,z)dy$$

or

$$\int_a^b dz \int_{\varphi_1(z)}^{\varphi_2(z)} dx \int_{\psi_1(x,z)}^{\psi_2(x,z)} f(x,y,z)dy$$

So, if the region V is regular in direction of all coordinate axes, six orders of integration are possible. The conversion of the iterated integral for one order of integration to the iterated integral for another order of integration is called the change of the order of integration.

The iterated integral has the most simple limits, if the region of integration is a rectangular box defined by $a \le x \le b$, $c \le y \le d$ and $p \le z \le q$. All the faces of that box are parallel to one of three coordinate planes.

If we choose x the outer variable, y the intermediate variable and z the inner variable, we compute

$$\int_a^b dx \int_c^d dy \int_p^q f(x,y,z)dy$$

and, of course, five more orders of integration are possible.

The iterated integral is the appropriate tool to compute the triple integral.
Theorem. If the function $f(x,y,z)$ is continuous in the closed regular region V, then

$$\iiint\limits_V f(x,y,z)dxdydz = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} dy \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz \qquad (5.4)$$

Example 2. Compute the triple integral

$$\iiint\limits_V xyzdxdydz$$

if the region V is bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x+y+z = 1$.

First three planes are the coordinate planes. The fourth plane passes three points $(1;0;0)$, $(0;1;0)$ and $(0;0;1)$. The intersection line of this plane and xy plane $z = 0$ is $x + y = 1$.

The projection of the region of integration onto the xy plane is the triangle, which is determined by equalities $0 \le x \le 1$ and $0 \le y \le 1 - x$. Since the region of integration is bounded by the plane $z = 0$ on the bottom and by $z = 1-x-y$ on the top, the region of integration is determined by $0 \le x \le 1$, $0 \le y \le 1 - x$ and $0 \le z \le 1 - x - y$. By the formula (5.4)

$$\iiint\limits_V xyzdxdydz = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} xyzdz$$

First we compute the inside integral

$$\int_0^{1-x-y} xyzdz = xy\frac{z^2}{2}\Big|_0^{1-x-y} = xy\frac{(1-x-y)^2}{2}$$

Next we integrate with respect to y

$$\int_0^{1-x} xy\frac{(1-x-y)^2}{2}dy = \frac{x}{2}\int_0^{1-x} y[(1-x)^2 - 2(1-x)y + y^2]dy$$

$$= \frac{x}{2}\left[(1-x)^2\frac{y^2}{2} - 2(1-x)\frac{y^3}{3} + \frac{y^4}{4}\right]\Big|_0^{1-x}$$

$$= \frac{x}{2}\left[\frac{(1-x)^4}{2} - \frac{2(1-x)^4}{3} + \frac{(1-x)^4}{4}\right] = \frac{x(1-x)^4}{24}$$

and finally

$$\frac{1}{24}\int_0^1 x(1-x)^4dx = -\frac{1}{24}\int_0^1 (-x)(1-x)^4dx$$

$$= \quad -\frac{1}{24}\int_0^1 (1-x-1)(1-x)^4dx = \frac{1}{24}\int_0^1 [(1-x)^5-(1-x)^4]d(1-x)$$

$$= \quad \frac{1}{24}\left[\frac{(1-x)^6}{6}-\frac{(1-x)^5}{5}\right]\Bigg|_0^1 = \frac{1}{24}(-\frac{1}{6}+\frac{1}{5}) = \frac{1}{720}$$

5.3 Change of variable in triple integral

Changing variables in triple integrals is nearly identical to changing variables in double integrals. We are going to change the variables in the triple integral

$$\iiint\limits_V f(x,y,z)dxdydz$$

over the region V in the xyz space. We use the transformation

$$\begin{cases} x = \varphi(u,v,w) \\ y = \psi(u,v,w) \\ z = \chi(u,v,w) \end{cases} \tag{5.5}$$

to transform the region V into the new region V' in the uvw space. We assume that the functions x, y and z of the variables u, v and w are one-valued and the system of equations (5.5) has unique solution for u, v and w. Then to any point in the region V' there is related one point in the region V and vice versa. In addition we assume that the functions (5.5) are continuous and they have continuous partial derivatives with respect to all three variables in the region V'.

The jacobian of this change of variables is the determinant

$$J = \begin{vmatrix} x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \\ x'_w & y'_w & z'_w \end{vmatrix} \tag{5.6}$$

and we can transform the triple integral over the region V into the triple integral over the region V' by the formula

$$\iiint\limits_V f(x,y,z)dxdydz = \iiint\limits_{V'} f(\varphi(u,v,w),\psi(u,v,w),\chi(u,v,w))|J|dudvdw \tag{5.7}$$

5.4 Triple integral in cylindrical coordinates

The cylindrical coordinates are really nothing more than an extension of polar coordinates into three dimensions leaving the z coordinate unchanged. For the given point $P(x, y, z)$ in the xyz space we denote P' the projection of this point onto xy plane. Denote by ρ the distance of P' from the origin and by φ the angle between the segment $P'O$ and x axis. Those φ and ρ are exactly the same as the polar coordinates in the two-dimensional case.

Definition. The cylindrical coordinates of the point P are called φ, ρ and z.

Since φ and ρ have in the xy plane the same meaning as the polar coordinates then the conversion formulas from the Cartesian coordinates into cylindrical coordinates are

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases} \tag{5.8}$$

Find the jacobian of this change of variables. By the formula (5.6) we get

$$J = \begin{vmatrix} x'_\varphi & y'_\varphi & z'_\varphi \\ x'_\rho & y'_\rho & z'_\rho \\ x'_z & y'_z & z'_z \end{vmatrix}$$

The variable z does not depend on φ and ρ, hence, $z'_\varphi = 0$ and $z'_\rho = 0$. The variables x and y does not depend on z, i.e. $x'_z = 0$ and $y'_z = 0$. Consequently,

$$J = \begin{vmatrix} -\rho\sin\varphi & \rho\cos\varphi & 0 \\ \cos\varphi & \sin\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding this determinant by the last column gives

$$J = \begin{vmatrix} -\rho\sin\varphi & \rho\cos\varphi \\ \cos\varphi & \sin\varphi \end{vmatrix} = -\rho\sin^2\varphi - \rho\cos^2\varphi = -\rho$$

Since ρ is a distance $|J| = \rho$.

Let V' be the region in cylindrical coordinates, which corresponds to the region V in Cartesian coordinates. By the general formula for change of variables in the triple integral (5.7) we obtain the formula to convert the triple integral in Cartesian coordinates into the triple integral in cylindrical coordinates

$$\iiint\limits_V f(x,y,z)dxdydz = \iiint\limits_{V'} f(\rho\cos\varphi, \rho\sin\varphi, z)\rho d\varphi d\rho dz \tag{5.9}$$

Supposing that the region V' in cylindrical coordinates is given by the inequalities $\alpha \leq \varphi \leq \beta$, $\rho_1(\varphi) \leq \rho \leq \rho_2(\varphi)$ and $z_1(\varphi,\rho) \leq z \leq z_2(\varphi,\rho)$, we can write by the formula (5.4)

$$\iiint\limits_V f(x,y,z)dxdydz = \int\limits_{\alpha}^{b} d\varphi \int\limits_{\rho_1(\varphi)}^{\rho_2(\varphi)} d\rho \int\limits_{z_1(\varphi,\rho)}^{z_2(\varphi,\rho)} f(\rho\cos\varphi, \rho\sin\varphi, z)\rho dz \tag{5.10}$$

Example 1. Convert $\int\limits_{-1}^{1} dy \int\limits_{0}^{\sqrt{1-y^2}} dx \int\limits_{x^2+y^2}^{\sqrt{x^2+y^2}} f(x,y,z)dz$ into an integral in cylindrical coordinates.

The ranges of the variables in Cartesian coordinates from this iterated integral are

$$\begin{gathered} -1 \leq y \leq 1 \\ 0 \leq x \leq \sqrt{1-y^2} \\ x^2+y^2 \leq z \leq \sqrt{x^2+y^2} \end{gathered}$$

The first two inequalities define the projection D of this region onto xy-plane, which is the half of the disk of radius 1 centered at the origin. The third equality determines that the region of integration is bounded by the paraboloid of rotation $z = x^2 + y^2$ on the bottom and by the cone $z = \sqrt{x^2+y^2}$ on the top.

In cylindrical coordinates the equation on the paraboloid of rotation converts to $z = \rho^2$ and the equation of the cone to $z = \rho$. So, the ranges for the region of integration in cylindrical coordinates are,

$$\begin{gathered} -\tfrac{\pi}{2} \leq \varphi \leq \tfrac{\pi}{2} \\ 0 \leq \rho \leq 1 \\ \rho^2 \leq z \leq \rho \end{gathered}$$

Now, by the formula (5.10) we write

$$\int_{-1}^{1} dy \int_{0}^{\sqrt{1-y^2}} dx \int_{x^2+y^2}^{\sqrt{x^2+y^2}} f(x,y,z)dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_{0}^{1} d\rho \int_{\rho^2}^{\rho} f(\rho\cos\rho, \rho\sin\varphi, z)\rho dz$$

Notice that the limits of integration are simpler in the cylindrical coordinates.

Example 2. Using the cylindrical coordinate, compute the triple integral

$$\int_{0}^{2} dx \int_{0}^{\sqrt{2x-x^2}} dy \int_{0}^{a} z\sqrt{x^2+y^2}dz$$

In Cartesian coordinates the region of integration is defined by the inequalities $0 \le x \le 2$, $0 \le y \le \sqrt{2x-x^2}$ and $0 \le z \le a$, i.e. bounded by the planes $x = 0$, $x = 2$, $y = 0$, $z = 0$ and $z = a$ and by the cylinder $y = \sqrt{2x-x^2}$. The generatrix of the cylinder is parallel to the z axis and the projection onto xy plane is the half circle $y = \sqrt{2x-x^2}$. This is the upper half of the circle $y^2 = 2x - x^2$ or $x^2 - 2x + y^2 = 0$, i.e.

$$(x-1)^2 + y^2 = 1$$

which is the circle of radius 1 centered at $(1;0)$.

Convert the integral given into the integral in cylindrical coordinates. The range of the angle φ in the projection of this region onto xy plane is $0 \le \varphi \le \dfrac{\pi}{2}$. Converting the equation of the cylinder $x^2 + y^2 = 2x$ into cylindrical coordinates gives $\rho^2\cos^2\varphi + \rho^2\sin^2\varphi = 2\rho\cos\varphi$ or $\rho = 2\cos\varphi$. Hence, the range for ρ is $0 \le \rho \le 2\cos\varphi$. We didn't convert the third coordinate z, thus, $0 \le z \le a$.

Converting the integrand into cylindrical coordinates gives

$$z\sqrt{\rho^2\cos^2\varphi + \rho^2\sin^2\varphi} = z\rho$$

Now, by the formula by (5.10)

$$\int_{0}^{2} dx \int_{0}^{\sqrt{2x-x^2}} dy \int_{0}^{a} z\sqrt{x^2+y^2}dz = \int_{0}^{\frac{\pi}{2}} d\varphi \int_{0}^{2\cos\varphi} d\rho \int_{0}^{a} z\rho \cdot \rho dz$$

The integration with respect to z gives

$$\int_0^a z\rho^2 dz = \rho^2 \frac{z^2}{2}\bigg|_0^a = \frac{a^2\rho^2}{2}$$

the integration with respect to ρ gives

$$\frac{a^2}{2}\int_0^{2\cos\varphi} \rho^2 d\varrho = \frac{a^2}{2}\frac{\rho^3}{3}\bigg|_0^{2\cos\varphi} = \frac{4a^2\cos^3\varphi}{3}$$

Finally, integrating with respect to φ, we get

$$\frac{4a^2}{3}\int_0^{\frac{\pi}{2}} \cos^3\varphi d\varphi = \frac{4a^2}{3}\int_0^{\frac{\pi}{2}} (1-\sin^2\varphi) d(\sin\varphi) = \frac{4a^2}{3}\left(\sin\varphi - \frac{\sin^3\varphi}{3}\right)\bigg|_0^{\frac{\pi}{2}} = \frac{8a^2}{9}$$

Finally we use the formula (5.3) to compute the volume of a solid.

Example 3 Compute the volume of solid bounded by the cone $z = \sqrt{x^2+y^2}$ and paraboloid of revolution $z = 2 - x^2 - y^2$.

First we find the intersection of these two surfaces. The equation of the cone can be converted to

$$z^2 = x^2 + y^2$$

and substituting x^2+y^2 into the equation of paraboloid we get $z = 2 - z^2$ or $z^2 + z - 2 = 0$.

This quadratic equation has two solutions $z_1 = 1$ and $z_2 = -2$. The second solution is impossible because of the equation of cone. Thus, these two surfaces intersect on the plane $z = 1$ and the intersection curve is the circle $x^2 + y^2 = 1$.

According to (5.3) the volume is

$$V = \iiint_V dxdydz$$

To evaluate this triple integral we use cylindrical coordinates. The projection of this solid onto xy-plane is the disk $x^2+y^2 \leq 1$. In cylindrical coordinates this disk is determined by inequalities $0 \leq \varphi \leq 2\pi$ and $0 \leq \rho \leq 1$. The surface on the top is paraboloid of revolution and the surface on the bottom is cone. In cylindrical coordinates $x^2 + y^2 = \rho^2$ thus, the equation of the cone is in cylindrical coordinates $z = \rho$ and the equation of the paraboloid is

$z = 2-\rho^2$. Consequently, in this region $\rho \le z \le 2-\rho^2$ and our triple integral is in cylindrical coordinates

$$V = \iiint\limits_V dxdydz = \int\limits_0^{2\pi} d\varphi \int\limits_0^1 d\rho \int\limits_\rho^{2-\rho^2} \rho dz$$

Integration with respect to z gives

$$\rho \cdot z\Big|_\rho^{2-\rho^2} = \rho(2-\rho^2-\rho) = 2\rho - \rho^3 - \rho^2$$

Integration with respect to ρ gives

$$\int\limits_0^1 (2\rho - \rho^3 - \rho^2)d\rho = \left(\rho^2 - \frac{\rho^4}{4} - \frac{\rho^3}{3}\right)\Big|_0^1 = \frac{5}{12}$$

and the volume of the solid is

$$V = \int\limits_0^{2\pi} \frac{5}{12} d\varphi = \frac{5}{12}\varphi\Big|_0^{2\pi} = \frac{5\pi}{6}$$