7 Surface integral

7.1 Surface integral of scalar fields

In mathematical analysis, a surface integral is a generalization of multiple integrals to integration over surfaces. It is like the double integral analog of the line integral. One may integrate over given surface scalar fields and vector fields. Let's start from the integration scalar fields over surface.

Suppose that the function of three variables $f(x, y, z)$ is defined on the surface S in the xyz axes.

Choose whatever partition of the surface S into n subsurfaces $\Delta\sigma_k$ $(1 \leq k \leq n)$, where $\Delta\sigma_k$ denotes the kth subsurface as well as its area.

On any of these subsurfaces we pick a random point $P_k(\xi_k; \eta_k; \zeta_k) \in \Delta\sigma_k$ and find the products

$$f(P_k)\Delta\sigma_k$$

Adding those products, we get the *integral sum* of the function $f(x, y, z)$ *over the surface* S

$$\sum_{k=1}^{n} f(P_k)\Delta\sigma_k$$

The greatest distance between the points on the subsurface is called the diameter of the subsurface $\operatorname{diam}\Delta\sigma_k$. Every subsurface has its own diameter. In general those diameters are different because we have the random partition of the surface S. Denote the greatest diameter by λ, i.e.

$$\lambda = \max_{1\leq k\leq n} \operatorname{diam}\Delta\sigma_k$$

Definition 1. If there exists the limit

$$\lim_{\lambda\to 0}\sum_{k=1}^{n} f(P_k)\Delta\sigma_k$$

and this limit does not depend on the partition of the surface S and does not depend on the choice of points P_k on the subsurfaces, then this limit is called the *surface integral with respect to area of surface* and denoted

$$\iint\limits_S f(x, y, z)d\sigma$$

By Definition 1

$$\iint_S f(x,y,z)d\sigma = \lim_{\lambda\to 0}\sum_{k=1}^{n} f(P_k)\Delta\sigma_k$$

Sometimes the surface integral with respect to area of surface is referred as the *surface integral of the scalar field.* The properties of the surface integral with respect to area of surface are familiar already. While formulating the properties, we use the term "surface integral"and "with respect to area of surface"will be omitted.

Property 1. The surface integral of the sum (difference) of two functions equals to the sum (difference) of surface integrals of these functions:

$$\iint_S [f(x,y,z) \pm g(x,y,z)]d\sigma = \iint_S f(x,y,z)d\sigma \pm \iint_S g(x,y,z)d\sigma$$

Property 2. The constant factor can be taken outside the surface integral, i.e. if c is a constant then

$$\iint_S cf(x,y,z)d\sigma = c\iint_S f(x,y,z)d\sigma$$

Property 3. If the surface is the unit of two surfaces, $S = S_1 \cup S_2$ and S_1 and S_2 have no common interior point, then

$$\iint_S f(x,y,z)d\sigma = \iint_{S_1} f(x,y,z)d\sigma + \iint_{S_2} f(x,y,z)d\sigma$$

Suppose the surface S is the graph of the function of two variables $z = z(x,y)$. Denote by D the projection of the surface S onto xy plane. The surface S is called *smooth* if the function $z(x,y)$ has continuous partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ in D.

The following theorem gives the formula to evaluate the surface integral with respect to area of surface.

Theorem. If the function $f(x,y,z)$ is continuous on the smooth surface S and D is the projection of S onto xy plane, then

$$\iint_S f(x,y,z)d\sigma = \iint_D f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy \quad (7.1)$$

Thus, in order to evaluate a surface integral we will substitute the equation of the surface in for z in the integrand and then add on the factor square root. After that the integral is a standard double integral and by this point we should be able to deal with that.

If the function $f(x, y, z) \equiv 1$ on the surface S, then the formula

$$\iint\limits_S d\sigma = \iint\limits_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy \tag{7.2}$$

gives us the area of the surface S.

Example 1. Evaluate $\iint\limits_S (x^2 + y^2 + z^2) d\sigma$, if S is the portion of the cone $z = \sqrt{x^2 + y^2}$, where $0 \leq z \leq 1$.

The plane $z = 1$ and the cone $z = \sqrt{x^2 + y^2}$ intersect along the circle

$$x^2 + y^2 = 1$$

The projection of the portion of the cone onto xy plane is the disk $x^2 + y^2 \leq 1$.

To apply the formula (7.1) we find

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

and

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} = \sqrt{2}$$

By the formula (7.1)

$$\iint\limits_S (x^2 + y^2 + z^2) d\sigma = \iint\limits_D (x^2 + y^2 + x^2 + y^2)\sqrt{2} dxdy = 2\sqrt{2} \iint\limits_D (x^2 + y^2) dxdy$$

The region of integration D in the double integral obtained is the disk of radius 1 centered at the origin. To compute this double integral we convert it into polar coordinates $x = \rho\cos\varphi$, $y = \rho\sin\varphi$. Then $x^2 + y^2 = \rho^2$ and $|J| = \rho$.

The region of integration in polar coordinates is determined by inequalities $0 \le \varphi \le 2\pi$ and $0 \le \rho \le 1$. Hence,

$$2\sqrt{2}\iint\limits_D (x^2+y^2)dxdy = 2\sqrt{2}\int\limits_0^{2\pi} d\varphi \int\limits_0^1 \rho^2\rho d\rho$$

First we compute the inside integral

$$\int\limits_0^1 \rho^3 d\rho = \frac{1}{4}$$

and finally the outside integral

$$2\sqrt{2}\int\limits_0^{2\pi} \frac{1}{4}d\varphi = \frac{\sqrt{2}}{2}\int\limits_0^{2\pi} d\varphi = \pi\sqrt{2}$$

Example 2. Compute the area of the portion of paraboloid of rotation $z = x^2 + y^2$ under the plane $z = 4$.

The projection D of the portion of paraboloid of rotation onto xy plane is the disk $x^2 + y^2 \le 4$ of radius 2 centered at the origin.we find

$$\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial z}{\partial y} = 2y$$

and

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial x}\right)^2} = \sqrt{1+4x^2+4y^2}$$

Thus, by the formula (7.2) the area of the portion of paraboloid of rotation is

$$\iint\limits_S d\sigma = \iint\limits_D \sqrt{1+4x^2+4y^2}dxdy$$

The double integral obtained we convert to polar coordinates $x = \rho\cos\varphi$, $y = \rho\sin\varphi$. Then $1 + 4x^2 + 4y^2 = 1 + 4\rho^2$ and $|J| = \rho$ and the region D is determined by $0 \le \varphi \le 2\pi$ and $0 \le \rho \le 2$. Therefore,

$$\iint\limits_D \sqrt{1+4x^2+4y^2}dxdy = \int\limits_0^{2\pi} d\varphi \int\limits_0^2 \sqrt{1+4\rho^2}\rho d\rho$$

To find the inside integral we use the equality of differentials $d(1+4\rho^2) = 8\rho d\rho$, which gives

$$\int_0^2 \sqrt{1+4\rho^2}\rho d\rho = \frac{1}{8}\int_0^2 \sqrt{1+4\rho^2}8\rho d\rho$$

$$= \frac{1}{8}\int_0^2 (1+4\rho^2)^{\frac{1}{2}} d(1+4\rho^2) = \frac{1}{8}\frac{(1+4\rho^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^2$$

$$= \frac{1}{12}(1+4\rho^2)\sqrt{1+4\rho^2}\Bigg|_0^2 = \frac{17\sqrt{17}-1}{12}$$

The outside integral, i.e. the area to be computed is

$$\frac{17\sqrt{17}-1}{12}\int_0^{2\pi} d\varphi = \frac{17\sqrt{17}-1}{12}\cdot 2\pi = \frac{\pi(17\sqrt{17}-1)}{6}$$

7.2 Surface integral with respect to coordinates

Suppose that S is a surface in the space and let $Z(x,y,z)$ be a function defined at all points of S. Choose a whatever partition of the surface S into n nonoverlapping subsurfaces $\Delta\sigma_k$ $(1 \leq k \leq n)$. In any of these subsurfaces we pick a random point $P_k(\xi_k;\eta_k;\zeta_k)$ and compute the value of function $Z(P_k)$. Let us denote by Δs_k the projection of $\Delta\sigma_k$ onto xy plane, where Δs_k denotes also the area of this projection. Next we find the products $Z(P_k)\Delta s_k$ and adding these products, we get the sum

$$\sum_{k=1}^{n} Z(P_k)\Delta s_k$$

which is called the integral sum of the function $Z(x,y,z)$ over the projection of surface S onto xy plane. Let $\operatorname{diam}\Delta s_k$ be the diameter of Δs_k. We have a random partition of the surface S, hence the diameters of these projections are different. Denote by λ the greatest diameter of the projections of subsurfaces $\Delta\sigma_k$, i.e.

$$\lambda = \max_{1\leq k\leq n} \operatorname{diam}\Delta s_k$$

Definition 1. If there exists the limit

$$\lim_{\lambda\to 0}\sum_{k=1}^{n} Z(P_k)\Delta s_k$$

and this limit does not depend on the partition of the surface S and it is independent on the choice of points P_k in the subsurfaces, then this limit is called the *surface integral* of the function $Z(x, y, z)$ *over the projection of the surface* onto xy plane and denoted

$$\iint_S Z(x, y, z)dxdy$$

Thus, by the definition

$$\iint_S Z(x, y, z)dxdy = \lim_{\lambda \to 0} \sum_{k=1}^{n} Z(P_k)\Delta s_k \tag{7.3}$$

Second, suppose that the function of three variables $Y(x, y, z)$ is defined at all points of the surface S and that $\Delta s'_k$ is the projection of $\Delta \sigma_k$ onto xz plane. Choosing again a random point $P_k \in \Delta \sigma_k$, we find the products $Y(P_k)\Delta s'_k$. The sum of these products

$$\sum_{k=1}^{n} Y(P_k)\Delta s'_k$$

is called the integral sum of the function $Y(x, y, z)$ over the projection of S onto xz plane.

Definition 2. If there exists the limit

$$\lim_{\lambda \to 0} \sum_{k=1}^{n} Y(P_k)\Delta s'_k$$

and this limit does not depend on the partition of the surface S and it is independent on the choice of points P_k in the subsurfaces, then this limit is called the *surface integral* of the function $Y(x, y, z)$ *over the projection of the surface* onto xz plane and denoted

$$\iint_S Y(x, y, z)dxdz$$

By Definition 2

$$\iint_S Y(x, y, z)dxdz = \lim_{\lambda \to 0} \sum_{k=1}^{n} Y(P_k)\Delta s'_k \tag{7.4}$$

Third, suppose that the function of three variables $X(x, y, z)$ is defined at all points of the surface S and $\Delta s_k''$ is the projection of $\Delta\sigma_k$ onto yz plane. We choose again random points $P_k \in \Delta\sigma_k$ and find the products $X(P_k)\Delta s_k''$. The sum

$$\sum_{k=1}^{n} X(P_k)\Delta s_k''$$

is called the integral sum of function $X(x, y, z)$ over the projection of S onto yz plane.

Definition 3. If there exists the limit

$$\lim_{\lambda\to 0}\sum_{k=1}^{n} X(P_k)\Delta s_k''$$

and this limit does not depend on the partition of the surface S and does not depend on the choice of points P_k in the subsurfaces, then this limit is called the *surface integral* of the function $X(x, y, z)$ *over the projection of the surface* onto yz plane and denoted

$$\iint_S X(x, y, z)dydz$$

By Definition 3

$$\iint_S X(x, y, z)dydz = \lim_{\lambda\to 0}\sum_{k=1}^{n} X(P_k)\Delta s_k'' \tag{7.5}$$

In general we define the surface integral over the projection of the vector function

$$\overrightarrow{F}(x, y, z) = (X(x, y, z); Y(x, y, z); Z(x, y, z))$$

as

$$\iint_S X(x, y, z)dydz + Y(x, y, z)dxdz + Z(x, y, z)dxdy \tag{7.6}$$

Remark. Sometimes the surface integral over the projection is also referred as the *surface integral of the vector field.*

7.3 Evaluation of surface integral over the projection

Consider the evaluation of the surface integral over the projection onto xy plane

$$\iint_S Z(x, y, z)dxdy$$

Suppose that the smooth surface S is a graph of the one-valued function of two variables $z = f(x, y)$. Since the function is one-valued, any line parallel to z axis cuts this surface exactly at one point.

Definition 1. A smooth surface S is said to be *two-sided*, if a normal vector is moved along any closed curve on the surface so that upon return to the starting point the direction of the normal is the same as it was originally. In the opposite case the surface is called one-sided.

A well known example of the one-sided surface is the *Möbius band*. It consists of a strip of paper with ends joined together to form a loop, but with one end given a half twist before the ends are joined.

For a two-sided surface we differ the upper and the lower side of the surface. The upper side of the surface is the side, where the normal vector forms an acute angle with z axis. The lower side of the surface is the side, where the normal vector forms an obtuse angle with z axis.

The evaluation of the surface integral over the projection depends on the side of the surface over which we integrate. If the function $Z(x, y, z)$ is continuous at any point of the smooth surface $z = f(x, y)$, then the surface integral over the projection onto xy plane is computed by the formula.

$$\iint\limits_S Z(x, y, z)dxdy = \pm \iint\limits_D Z(x, y, f(x, y))dxdy \tag{7.7}$$

On the right side of this formula is a standard double integral, where D denotes the projection of the surface S onto xy plane. Using this formula, we choose the sign "$+$", if we integrate over the upper side of surface and we choose the sign "$-$", if we integrate over the lower side of the surface. So, for any problem there has to be said over which side of the surface we need to integrate.

If the function $Y(x, y, z)$ is continuous at any point of the smooth surface $y = g(x, z)$, then the surface integral over the projection onto xz plane is computed by the formula

$$\iint\limits_{S} Y(x,y,z)dxdz = \pm \iint\limits_{D'} Y(x,g(x,z),z)dxdz \tag{7.8}$$

In this formula D' denotes the projection of S onto xz plane and the choice of the sign $+$ or $-$ depends on over which side of the surface the integration is carried out (i.e. does the normal of the surface forms with y axis acute or obtuse angle).

If the function $X(x,y,z)$ is continuous at any point of the smooth surface $x = h(y,z)$, then the surface integral over the projection onto yz plane is computed by the formula

$$\iint\limits_{S} X(x,y,z)dydz = \pm \iint\limits_{D''} X(h(y,z),y,z)dydz \tag{7.9}$$

Here D'' denotes the projection of S onto yz plane and the choice of the sign $+$ or $-$ depends on over which side of the surface the integration is carried out (i.e. does the normal of the surface forms with x axis acute or obtuse angle).

Example. Compute the surface integral

$$\iint\limits_{S} z^2 dxdy$$

where S is the upper side of the portion of cone $z = \sqrt{x^2+y^2}$ between the planes $z = 0$ and $z = 1$.

This portion of cone is sketched in Figure 8.8. The projection D onto xy plane of this portion of cone is the disk $x^2 + y^2 \leq 1$. Hence by (7.7)

$$\iint\limits_{\sigma} z^2 dxdy = \iint\limits_{D} (x^2+y^2)dxdy$$

Since the region of integration is the disk, we convert the double integral into polar coordinates. For this disk $0 \leq \varphi \leq 2\pi$ and $0 \leq \rho \leq 1$, thus,

$$\iint\limits_{D} (x^2+y^2)dxdy = \int\limits_0^{2\pi} d\varphi \int\limits_0^1 \rho^2 \cdot \rho d\rho$$

Now we compute

$$\int\limits_0^1 \rho^3 d\rho = \left.\frac{\rho^4}{4}\right|_0^1 = \frac{1}{4}$$

and

$$\int_0^{2\pi} \frac{1}{4} d\varphi = \frac{1}{4} \cdot 2\pi = \frac{\pi}{2}$$