8 Series

8.1 Series. Sum of series

The series is an infinite sum

$$u_1 + u_2 + \ \dots\ + u_k + \ \dots\ = \sum_{k=1}^{\infty} u_k \tag{8.1}$$

The addends in this infinite sum are called the terms of the series and u_k is called the *general term.* If we assign to k some natural number, we get the related term of the series. In (8.1) the k is called the index of summation and note that the letter we use to represent the index can be any integer variable i, j, l, m, n, $\dots$. The first index is 1 for convenience, actually it can be any integer. We can write (8.1) as

$$\sum_{k=1}^{\infty} u_k = \sum_{k=0}^{\infty} u_{k+1} = \sum_{k=2}^{\infty} u_{k-1} = \dots$$

A *number series* is the series, whose terms are numbers. In our course we consider the series of real numbers. A *functional series* is the series, whose terms are functions of the variable x, i.e. $u_k = u_k(x)$, $k = 1,\ 2,\ \dots$.

A geometric series is the series

$$a + aq + aq^2 + \ \dots\ + aq^k + \ \dots\ = \sum_{k=0}^{\infty} aq^k \tag{8.2}$$

where each successive term is produced by multiplying the previous term by a constant number q (called the *common ratio* in this context).

The harmonic series is the series

$$1 + \frac{1}{2} + \frac{1}{3} + \ \dots\ + \frac{1}{k} + \ \dots\ = \sum_{k=1}^{\infty} \frac{1}{k} \tag{8.3}$$

The sum of the first n terms

$$S_n = \sum_{k=1}^{n} u_k$$

is called the nth partial sum of the series. The partial sums

$$S_1 = u_1$$
$$S_2 = u_1 + u_2$$

....

$$S_n = u_1 + u_2 + \ \dots \ + u_n$$

....

define the sequence of partial sums

$$S_1, \ S_2, \ \dots, \ S_n, \ \dots \tag{8.4}$$

Definition. A series (8.1) is said to *converge* or to *be convergent* when the sequence (8.4) of partial sums has a finite limit. If the limit of (8.4) is infinite or does not exist, the series is said to *diverge* or to *be divergent.* When the limit of partial sums

$$\lim_{n\to\infty} S_n = S$$

exists, it is called the *sum of the series* and one writes

$$S = \sum_{k=1}^{\infty} u_k$$

It is important not to get sequences and series confused! A sequence is a list of numbers written in a specific order while an infinite series is a limit of a sequence and hence, if it exists will be a single value.

Example 1. The sum of the first n terms, i.e. the $n-1$st partial sum of the geometric series is

$$S_{n-1} = \sum_{k=0}^{n-1} aq^k = \frac{a(1-q^n)}{1-q}$$

If $|q| < 1$, then

$$\lim_{n\to\infty} q^n = 0$$

thus,

$$\lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} \frac{a(1-q^n)}{1-q} = \lim_{n\to\infty} \frac{a}{1-q} - \lim_{n\to\infty} \frac{aq^n}{1-q} = \frac{a}{1-q}$$

So, if $|q| < 1$, then the geometric series converges and the sum is

$$S = \frac{a}{1-q}$$

If $q > 1$, then

$$\lim_{n\to\infty} q^n = \infty$$

therefore,

$$\lim_{n\to\infty} S_{n-1} = \infty$$

and the geometric series is divergent If $q < -1$, then $\lim_{n\to\infty} q^n$ does not exist and hence, $\lim_{n\to\infty} S_{n-1}$ does not exist and the geometric series is divergent. If $q = 1$, then the $n-1$st partial sum

$$S_n = \sum_{k=0}^{n-1} aq^k = \sum_{k=0}^{n-1} a = na$$

and the limit $\lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} = na = \infty$. If $q = -1$, then the $S_0 = a$, $S_1 = a - a = 0$, $S_2 = a - a + a = a$, $S_3 = a - a + a - a = 0$, ... We obtain the sequence of partial sums

$$a,\ 0,\ a,\ 0,\ \ldots$$

which has no limit. Therefore, for $q = \pm 1$ the geometric series is divergent.

Conclusion. If $|q| < 1$, then the geometric series (8.2) converges and if $|q| \geq 1$ then the geometric series diverges.

Example 2. To find the nth partial sum S_n of the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

we use the partial fractions decomposition

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

We obtain

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)}$$
$$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

The limit of this sequence, i.e. the sum of this series

$$S = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$$

If we ignore the first term the remaining terms will also be a series that will start at $k = 2$ instead of $k = 1$ So, we can rewrite the original series (8.1) as follows,

$$\sum_{k=1}^{\infty} u_k = u_1 + \sum_{k=2}^{\infty} u_k$$

We say that we've stripped out the first term. We could have stripped out the first two terms

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \sum_{k=3}^{\infty} u_k$$

and first any number of terms respectively,

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \ldots + u_m + \sum_{k=m+1}^{\infty} u_k = \sum_{k=1}^{m} u_k + \sum_{k=m+1}^{\infty} u_k$$

The first sum on the right side of this equality is the mth partial

$$\sum_{k=1}^{m} u_k$$

sum of series (8.1). This is a finite sum, which is always finite. Assuming that $n > m$, we can write the nth partial sum

$$\sum_{k=1}^{n} u_k = \sum_{k=1}^{m} u_k + \sum_{k=m+1}^{n} u_k$$

or

$$S_n = S_m + S_{n-m}$$

where

$$S_{n-m} = \sum_{k=m+1}^{n} u_k$$

Now, if S_n has the finite limit as $n \to \infty$, then S_{n-m} must have also the finite limit. Conversely, if S_{n-m} has the finite limit as $n \to \infty$, then adding the finite sum S_m leaves the limit finite.

Similarly, S_n has the infinite limit or does not have the limit if and only if S_{n-m} has also the infinite limit or has no limit.

Conclusion. Stripping out the finite number of terms from the beginning of the series leaves the convergent series convergent and divergent series divergent. As well, adding the finite number of terms to the beginning of the series does not make the convergent series divergent and does not make the divergent series convergent.

8.2 Necessary condition for convergence of series

Suppose that the series (8.1) converges to the sum S, i.e.

$$\lim_{n\to\infty} S_n = S$$

The nth partial sum can be written

$$S_n = \sum_{k=1}^{n} u_k = \sum_{k=1}^{n-1} u_k + u_n$$

or

$$S_n = S_{n-1} + u_n$$

hence,

$$u_n = S_n - S_{n-1}$$

The convergence of the series gives, since if $n \to \infty$ then $n - 1 \to \infty$,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$$

We have proved an essential theorem, so called *necessary condition for the convergence of the series.*

Theorem 1. If the series (8.1) converges, then the limit of the general term

$$\lim_{n\to\infty} u_n = 0 \tag{8.5}$$

This theorem gives us a requirement for convergence but not a guarantee of convergence. In other words, the converse is not true. If $\lim\limits_{n\to\infty} u_n = 0$ the series may actually diverge. For example, the limit of the general term of the harmonic series (8.3)

$$\lim_{k\to\infty} \frac{1}{k} = 0$$

but the harmonic series is divergent. It will be a couple of subsections before we can prove this, so at this point the reader has just to believe this and know that it's possible to prove the divergence.

In order for a series to converge the series terms must go to zero in the limit. If the series terms do not go to zero in the limit then there is no way the series can converge since this would contradict the theorem, i.e. there holds.

Conclusion (the divergence test). If $\lim\limits_{n\to\infty} u_n \neq 0$ then the series (8.1) diverges.

For example the series

$$\sum_{k=1}^{\infty} 1$$

is divergent because the limit of the constant term is that constant,

$$\lim_{k\to\infty} 1 = 1 \neq 0$$

8.3 Convergence tests of positive series

In Mathematical analysis there exist a lot of tests that give us the possibility to decide whether the series converges or diverges. In this subsection we are going to consider the positive series, i.e. the series (8.1), whose all terms are positive:

$$u_k \geq 0, \quad k = 1,\ 2,\ \ldots$$

8.3.1 Comparison test

The nth partial sum of the series (8.1) is

$$S_n = S_{n-1} + u_n$$

Since for any index n $u_n \geq 0$, then

$$S_n \geq S_{n-1}$$

that means, the sequence of partial sums of the positive series is monotonically increasing. We had the theorem in Mathematical analysis I, which stated that any bounded monotonically increasing sequence has the finite limit. So, if we have succeeded to prove that the sequence of the partial sums of the positive series is bounded, we have proved the existence of the finite limit of the sequence of partial sums, that is, we have proved the convergence of the positive series.

The sequence

$$S_1,\ S_2,\ \ldots,\ S_n,\ \ldots$$

has the finite limit means by the definition of the limit that for any $\varepsilon > 0$ there exists the index $N > 0$ such that for all $n \geq N$

$$|S_n - S| < \varepsilon$$

This inequality is identical to the inequalities

$$-\varepsilon < S_n - S < \varepsilon$$

or

$$S - \varepsilon < S_n < S + \varepsilon$$

which means the sequence is bounded. We have proved the following theorem.

Theorem 1. The positive series (8.1) is convergent if and only if the sequence of its partial sums is bounded.

Suppose that we have another positive series

$$\sum_{k=1}^{\infty} v_k \tag{8.6}$$

and we know whether it converges or diverges. For instance we know that the geometric series (8.2) converges if $|q| < 1$ and diverges if $|q| \geq 1$. We know that the harmonic series is divergent and we know that

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

is convergent.

Theorem 2 (the comparison test). 1) If for any $k = 1,\ 2,\ 3,\ \ldots$

$$u_k \leq v_k$$

then the convergence of the series (8.6) yields the convergence of the series (8.1).

2) If for any $k = 1,\ 2,\ 3,\ \ldots$

$$u_k \geq v_k$$

then the divergence of the series (8.6) yields the divergence of the series (8.1).

Example 1. Prove that the series

$$1 + \frac{1}{4} + \frac{1}{9} + \ \ldots\ + \frac{1}{k^2} + \ \ldots\ = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges.

We know that the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=2}^{\infty} \frac{1}{(k-1)k}$$

converges. For any $k = 2,\ 3,\ \ldots$ it is obvious that

$$\frac{1}{k^2} < \frac{1}{(k-1)k}$$

and by Theorem 2 the series

$$\sum_{k=2}^{\infty} \frac{1}{k^2}$$

converges. Adding the term 1 to the beginning of the series preserves the convergence.

Example 2. Prove that the series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ \dots \ + \frac{1}{\sqrt{k}} + \ \dots \ = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

diverges.

For any $k \geq 1$ there holds the inequality $\sqrt{k} \leq k$ hence,

$$\frac{1}{\sqrt{k}} > \frac{1}{k}$$

The harmonic series (8.3) diverges thus, by Theorem 2 the series given diverges also.

8.3.2 D'Alembert's test

Sometimes the D'Alembert's test is referred as the ratio test. We consider again the positive series (8.1).

Theorem (D'Alembert's test). Suppose there exists the limit

$$\lim_{k\to\infty} \frac{u_{k+1}}{u_k} = D$$

1) If $D < 1$, then the series (8.1) converges.

2) If $D > 1$, then series (8.1) diverges.

3) If $D = 1$, then this test us inconclusive, because there exist both convergent and divergent series that satisfy this case.

Example 1. Does the series $\sum\limits_{k=1}^{\infty} \frac{1}{k!}$ converge or diverge?

The ratio of two consecutive terms $u_{k+1} = \frac{1}{(k+1)!}$ and $u_k = \frac{1}{k!}$ is

$$\frac{u_{k+1}}{u_k} = \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \frac{k!}{(k+1)k!} = \frac{1}{k+1}$$

and the limit of this ratio

$$D = \lim_{k\to\infty} \frac{1}{k+1} = 0$$

Since $D = 0$, this series converges by the D'Alembert's test.

Example 2. Does the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converge or diverge?

Compute the limit

$$D = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2}{(k+1)^2} = 1$$

Since $D = 1$, the D'Alembert's test is inconclusive, but we know that by the comparison test that this series converges.

Example 3. Does the series $\sum_{k=1}^{\infty} \frac{1}{k}$ converge or diverge?

For the harmonic series we have

$$D = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{\frac{1}{k+1}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

so, the harmonic series cannot be handled by the D'Alembert's test, but we know that the series diverges.

8.3.3 Cauchy test

Cauchy test is also known as root test of convergence of a series. Let us consider the positive series (8.1) again.

Theorem (Cauchy test). Suppose there exists the limit

$$\lim_{k\to\infty} \sqrt[k]{u_k} = C$$

1) If $C < 1$, then the series (8.1) converges.
2) If $C > 1$, then series (8.1) diverges.
3) If $C = 1$, then this test us inconclusive.

Example 1. Determine if the series

$$\sum_{k=1}^{\infty} \frac{k^2}{2^k}$$

is convergent or divergent?

To use the Cauchy test we find $\sqrt[k]{u_k} = \frac{\sqrt[k]{k^2}}{2}$ and evaluate the limit

$$C = \lim_{k\to\infty} \sqrt[k]{u_k} = \lim_{k\to\infty} \frac{\sqrt[k]{k^2}}{2} = \frac{1}{2} \lim_{k\to\infty} k^{\frac{2}{k}}$$

Since we have the indeterminate form ∞^0, we apply the L'Hospital's rule

$$\lim_{k\to\infty} \ln k^{\frac{2}{k}} = \lim_{k\to\infty} \frac{2}{k}\ln k$$
$$= \lim_{k\to\infty} \frac{(2\ln k)'}{k'} = \lim_{k\to\infty} \frac{2}{k} = 0$$

and

$$C = \frac{1}{2}e^0 = \frac{1}{2} < 1$$

So, by the Cauchy test the series is convergent.

Example 2. Determine if the series

$$\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^{k^2}$$

is convergent or divergent?

The kth root of the general term is

$$\sqrt[k]{u_k} = \sqrt[k]{\left(1+\frac{1}{k}\right)^{k^2}} = \left(1+\frac{1}{k}\right)^{k}$$

and the limit

$$C = \lim_{k\to\infty} \sqrt[k]{u_k} = \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^{k} = e > 1$$

Hence, by the Cauchy test the series is divergent.

8.3.4 Integral test

Let us consider the a positive series (8.1) once more.

Theorem 5 (Integral test). Suppose $u(x)$ is a continuous positive decreasing on interval $[1;\infty)$ function, whose values for the integer arguments are the terms of series (8.1), i.e. $u(k) = u_k$. Then

1) if the improper integral (8.1) $\int\limits_1^{\infty} u(x)dx$ is convergent so is the series (8.1);

1) if the improper integral (8.1) $\int\limits_1^{\infty} u(x)dx$ is divergent so is the series (8.1).

Example 4. Prove that the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

diverges.

To apply the integral test we define the decreasing function $u(x) = \dfrac{1}{x}$, whose values for the integer arguments $x = k$ are

$$u_k = u(k) = \frac{1}{k}$$

The improper integral is divergent because

$$\int_1^{\infty} \frac{dx}{x} = \lim_{N\to\infty} \int_1^{N} \frac{dx}{x} = \lim_{N\to\infty} \ln|x| \Big|_1^N = \lim_{N\to\infty} \ln N = \infty$$

By the Integral test the harmonic series is divergent.

8.4 Alternating series. Leibnitz's test.

The last tests that we looked at for series convergence have required that all the terms in the series be positive. The test that we are going to look into in this subsection will be a test for alternating series. An alternating series is any series

$$u_1 - u_2 + u_3 - u_4 + \ \dots = \sum_{k=1}^{\infty} (-1)^{k+1} u_k \tag{8.7}$$

or

$$-u_1 + u_2 - u_3 + u_4 - \ \dots = \sum_{k=1}^{\infty} (-1)^{k} u_k$$

where $u_k > 0, \;\; k = 1,\ 2,\ \dots$

The second alternating series we can write

$$\sum_{k=1}^{\infty} (-1)^{k} u_k = -\sum_{k=1}^{\infty} (-1)^{k+1} u_k$$

therefore, it's enough to look at for convergence of the series (8.7).

Theorem 1. (Leibnitz's test) If

1) $u_k > u_{k+1}$, $k = 1,\ 2,\ \dots$ and

2) $\lim\limits_{k\to\infty} u_k = 0$, then the alternating series (8.7) converges.

Example. For the alternating harmonic series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$$

both of the assumptions of the theorem hold because

$$1>\frac{1}{2}>\ldots>\frac{1}{k}>\frac{1}{k+1}>\ldots$$

and

$$\lim_{k\to\infty}\frac{1}{k}=0$$

Hence, this series is convergent.

8.5 Absolute and conditional convergence

In this subsection we assume that the terms of the series

$$\sum_{k=1}^{\infty}u_k \tag{8.8}$$

can have whatever signs.

Definition 1. The series (8.8) is called *absolutely convergent* if the series

$$|u_1|+|u_2|+|u_3|+\ldots=\sum_{k=1}^{\infty}|u_k|$$

is convergent.

Theorem 1. If the series (8.8) is absolutely convergent then it is also convergent.

Proof. The definition of the absolute value

$$|u_k|=\begin{cases} u_k, & \text{if } u_k\geq 0\\ -u_k, & \text{if } u_k<0\end{cases}$$

gives us that

$$0\leq u_k+|u_k|\leq 2|u_k|$$

Since we are assuming that

$$\sum_{k=1}^{\infty}|u_k|$$

is convergent then

$$\sum_{k=1}^{\infty} 2|u_k| = 2\sum_{k=1}^{\infty} |u_k|$$

is also convergent because 2 times finite value will still be finite. The comparison test gives us that

$$\sum_{k=1}^{\infty} (u_k + |u_k|)$$

is also a convergent series. Now the series (8.8)

$$\sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} (u_k + |u_k| - |u_k|) = \sum_{k=1}^{\infty} (u_k + |u_k|) - \sum_{k=1}^{\infty} |u_k|$$

is the difference of two convergent series, i.e. convergent.

By Theorem 1 series that are absolutely convergent are guaranteed to be convergent. However, series that are convergent may or may not be absolutely convergent.

Definition 2. The series (8.8) which is convergent but not absolutely convergent is called *conditionally convergent.*

Example 1. Alternating harmonic series

$$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k}$$

is convergent by Leibnitz's test, but the series

$$\sum_{k=1}^{\infty} \left|(-1)^{k+1}\frac{1}{k}\right| = \sum_{k=1}^{\infty} \frac{1}{k}$$

is the harmonic series. By Integral test the harmonic series diverges hence, alternating harmonic series is a conditionally convergent series.

Example 2. Determine if the series $\sum\limits_{k=1}^{\infty} \dfrac{\sin k}{k^2}$ is absolutely convergent, conditionally convergent or divergent.

Notice that this is not an alternating series. Since $|\sin k| \leq 1$ for any integer k, then

$$\left|\frac{\sin k}{k^2}\right| \leq \frac{1}{k^2}$$

We know that the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges hence, by Comparison test the series

$$\sum_{k=1}^{\infty} \left| \frac{\sin k}{k^2} \right|$$

converges, i.e. the series $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ is absolutely convergent and Theorem 1 guarantees its convergence.

While the convergence of the positive series takes place because of the terms are decreasing with the sufficient speed, then the conditional convergence happens because the terms reduce each other.

8.6 Power series

A series of functions is the series, whose terms are the functions of some variable, suppose x

$$\sum_{k=1}^{\infty} u_k(x) \tag{8.9}$$

If we assign to the variable x a certain value x_0 that is in domains of all u_k and substitute it into all these functions, we have the numerical values $u_k(x_0)$, i.e for $x = x_0$ the series (8.9) is a number series.

Example. Let's examine the series of functions

$$1 + x + x^2 + \ldots + x^k + \ldots = \sum_{k=0}^{\infty} x^k \tag{8.10}$$

If the variable x has the value $x = \frac{1}{2}$, we get the geometric series

$$\sum_{k=0}^{\infty} \frac{1}{2^k}$$

which is convergent, because the common ratio is $\frac{1}{2}$.

Assigning to the variable x the value $x = 1$, we get the number series

$$1 + 1 + 1 + \ldots$$

which diverges by Divergence test. Assigning to the variable x the value $x = -1$, we get the divergent number series

$$1 - 1 + 1 - \ldots + (-1)^k + \ldots$$

Assigning to the variable x the some value $x_0 > 1$, we obtain the number series with general term

$$u_k(x_0) = x_0^k$$

which diverges by Divergence test because

$$\lim_{k \to \infty} x_0^k = \infty$$

Assigning to the variable x the some value $x_0 < -1$, we obtain the number series which diverges by Divergence test because the general term has no limit.

It has turned out that for some values of the variable x the series of functions converges and for other values it diverges.

The partial sums of the series of functions (8.9)

$$S_n(x) = \sum_{k=1}^{n} u_k(x)$$

are also functions of the variable x and define a sequence of functions

$$S_1(x),\ S_2(x),\ \ldots,\ S_n(x),\ \ldots \tag{8.11}$$

Definition. The set X of the values of argument x for which the sequence of partial sums (8.11) is convergent, i.e. there exists the (finite) limit

$$S(x) = \lim_{n \to \infty} S_n(x) \tag{8.12}$$

is called the *region of convergence* of the series of functions (8.9).

It is said that $S(x)$ is the sum of the series (8.9) and one writes

$$S(x) = \sum_{k=1}^{\infty} u_k(x)$$

Power series is a series of power functions

$$\sum_{k=0}^{\infty} c_k x^k \tag{8.13}$$

or in general

$$\sum_{k=0}^{\infty} c_k (x-a)^k \tag{8.14}$$

where the numbers c_k are called the coefficients of the series.

The examination of the properties of those series is very similar therefore, we restrict ourselves with series (8.13).

Example 1. The series

$$1 + x + x^2 + \ \dots \ + x^k + \ \dots = \sum_{k=0}^{\infty} x^k$$

is a geometric series for any value of x. This series converges if $|x| < 1$. Hence, the region of convergence of this series is open interval $X = (-1; 1)$ and the sum of this series in this interval is

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \tag{8.15}$$

It turns out that the regions of convergence of power series have such a simple structure.

Theorem 1 (Abel's theorem). If the power series (8.13) converges for some value of x_0, then this series converges absolutely for any value of $|x| < |x_0|$.

Conversely, if the power series (8.13) diverges for some value of x_0, then this series diverges for any value of $|x| > |x_0|$.

According to Abel's theorem there exists a real number R such that for $|x| < R$ the series (8.13) converges and for $|x| > R$ diverges. This real number R is called the *radius of convergence* of the series (8.13) and the interval $(-R; R)$ the *interval of convergence* of this series.

Remark. At the endpoints $x = R$ and $x = -R$ of the interval of convergence the series (8.13) may converge and may diverge. Therefore, to completely identify the interval of convergence all that we have to do is determine if the power series will converge for $x = R$ or $x = -R$. If the power series converges for one or both of these values then we'll need to include those in the interval of convergence.

There are a lot of possibilities to determine the radius of convergence of power series (8.13). One of these possibilities is to use the formula.

$$R = \lim_{k\to\infty} \left| \frac{c_k}{c_{k+1}} \right| \tag{8.16}$$

Example. Find the intervals of convergence of power series

$$\sum_{k=1}^{\infty} x^k$$

$$\sum_{k=1}^{\infty} \frac{x^k}{k}$$

and

$$\sum_{k=1}^{\infty} \frac{x^k}{k^2}$$

The radius of convergence is 1 for all of three series. The coefficient of the first series are $c_k = 1$ hence,

$$R = \lim_{k \to \infty} \frac{1}{1} = 1$$

The coefficients of the second series are $c_k = \frac{1}{k}$ and

$$R = \lim_{k \to \infty} \frac{k+1}{k} = 1$$

The coefficients of the third series are $c_k = \frac{1}{k^2}$ and

$$R = \lim_{k \to \infty} \frac{(k+1)^2}{k^2} = 1$$

thus, all three series are convergent if $-1 < x < 1$ and diverges if $|x| > 1$. Determine if these series will converge for $x = 1$ or $x = -1$.

The general term of the first series at the right endpoint is $1^k = 1$, whose limit $1 \neq 0$ hence, the series diverges. At the left endpoint the general term is $(-1)^k$, which has no limit as $k \to \infty$, i.e. the series diverges again and the interval of convergence of the first series is $(-1; 1)$

The general term of the second series at the right endpoint is $\frac{1}{k}$ hence, the second series is at the right endpoint the harmonic series, which is divergent. At the left endpoint the general term is $\frac{(-1)^k}{k}$, i.e. the second series is at the left endpoint the alternating harmonic series, which converges by Leibnitz's test. Thus, the interval of convergence of the second series is $[-1; 1)$.

The general term of the second series at the right endpoint is $\frac{1}{k^2}$ and at the left endpoint $\frac{(-1)^k}{k^2}$. The absolute value of both of these is $\frac{1}{k^2}$. By Example 1 of subsection 8.3 the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges thus, the third series converges at both endpoints and the interval of convergence is $[-1; 1]$.

Inside the interval of convergence of power series it's possible to prove.

Conclusion 1. If the radius of convergence of the power series (8.13) is R, then the sum of this series is continuous on any interval $[a; b] \subset (-R; R)$.

Conclusion 2. If the radius of convergence of the power series (8.13) is R, then this series can be integrated term by term on any interval $[a; b] \subset (-R; R)$.

Conclusion 3. If the radius of convergence of the power series (8.13) is R, then this series can be differentiated term by term on any interval $[a; b] \subset (-R; R)$.

Now, using the sum of the geometric series (8.15) and conclusions 2 and 3, we can find the power series expansions for many functions.

Example 1. Multiplying both sides of (8.15) by x gives

$$\frac{x}{1-x} = x \cdot \sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} x^{k+1}$$

and the radius of convergence is still 1. It's easy to verify that

$$\left(\frac{x}{1-x}\right)' = \frac{1}{(1-x)^2}$$

and using the term by term differentiation we get the power series expansion of this derivative

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (x^{k+1})' = \sum_{k=0}^{\infty} (k+1)x^k$$

and the radius of convergence of the series obtained is 1 again.

Example 2. If we substitute in (8.15) the variable x by $-x^2$, we get

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

and this series converges if $|-x^2| < 1$, which is equivalent to $|x| < 1$.

Since

$$\arctan x = \int_0^x \frac{dx}{1+x^2}$$

, we obtain the power series of arc tangent function integrating the last series term by term in limits from 0 to x provided $|x| < 1$.

$$\arctan x = \sum_{k=0}^{\infty} (-1)^k \int_0^x x^{2k} dx = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

and the radius of convergence is 1 hence, the interval of convergence is $(-1; 1)$.

At the left endpoint of the interval of convergence we get the series

$$\sum_{k=0}^{\infty} (-1)^k \frac{(-1)^{2k+1}}{2k+1} = -\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

and at the right endpoint

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

Both series obtained are the alternating series, which converge by the Leibnitz's test and therefore, the interval of convergence of the series obtained is $[-1; 1]$.

So, it may happen that the series obtained as the result of term by term integration converges at one or both of the endpoints, despite of the initial series diverges at the endpoints.

8.7 Taylor's and Maclaurin's series

Suppose that the function $f(x)$ is differentiable infinitely many times in the neighborhood of a. If the coefficients c_k of the power series

$$\sum_{k=0}^{\infty} c_k (x-a)^k$$

are computed by the formula

$$c_k = \frac{f^{(k)}(a)}{k!} \tag{8.17}$$

then these coefficients are called Taylor's coefficients and the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k \tag{8.18}$$

is called Taylor's series of the function $f(x)$ in the neighborhood of a or Taylor's series of the function $f(x)$ in powers $x - a$. The nth partial sum of this series is the Taylor's polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

By Taylor's formula the function $f(x)$ can be represented as

$$f(x) = P_n(x) + R_n(x)$$

that is the sum of the Taylor's polynomial and the remainder.

We know that Lagrange form of the remainder of the Taylor's formula is

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(a + \Theta(x-a))$$

where $0 < \Theta < 1$

If

$$\lim_{n\to\infty} R_n(x) = 0$$

then

$$\lim_{n\to\infty} P_n(x) = f(x)$$

which means that the sequence of partial sums of Taylor's series converges to the function $f(x)$.

Therefore, the series (8.20) represents the function $f(x)$ if and only if the limit of the remainder equals to 0. If $\lim\limits_{n\to\infty} R_n(x) \neq 0$, then the Taylor's series of the function $f(x)$ may still converge but it does not represent the function $f(x)$.

Taylor's series in the neighborhood of $a = 0$, i.e. Taylor's series in powers x

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \tag{8.19}$$

is called Maclaurin's series.

8.8 Maclaurin's series of functions e^x, $\sin x$ and $\cos x$

In Mathematical analysis I we have proved that Maclaurin's formula of nth degree of the exponential function e^x is

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots + \frac{1}{n!}x^n + R_n(x)$$

and that the limit of the remainder

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} e^{\Theta x} = 0$$

for each $x \in \mathbb{R}$ and for $0 < \theta < 1$. Consequently, Maclaurin's series represents the function e^x for every real x, i.e.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Also it has been proved that Maclaurin's formula of $2n + 1$st degree of the sine function $\sin x$ is

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + R_{2n+1}(x)$$

whose remainder is

$$R_{2n+1}(x) = \frac{x^{2n+2}}{(2n+2)!} \sin\left(\Theta x + (n+1)\pi\right)$$

Since for every $x \in \mathbb{R}$ and for $0 < \theta < 1$

$$\lim_{n \to \infty} R_{2n+1}(x) = 0$$

Maclaurin's series represents the function $\sin x$ for every real x:

$$\sin x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots$$

As well it has been proved that Maclaurin's formula of $2n$th degree of the cosine function $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + R_{2n}(x)$$

and the remainder

$$R_{2n}(x) = \frac{x^{2n+1}}{(2n+1)!} \cos\left(\Theta x + (2n+1)\frac{\pi}{2}\right)$$

Again, for every $x \in \mathbb{R}$ and for $0 < \theta < 1$

$$\lim_{n \to \infty} R_{2n}(x) = 0$$

hence, Maclaurin's series represents the function $\cos x$ for every real x:

$$\cos x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots$$

8.9 Fourier series of 2π-periodic functions

For an infinitely many times differentiable function $f(x)$ Maclaurin's series expansion is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \tag{8.20}$$

Here we have expanded the function $f(x)$ with respect to *system of power fuctions*

$$\{1,\ x, x^2,\ \ldots\}$$

Another system of functions is the *system of trigonometric functions*

$$\{1;\ \sin x;\ \cos x;\ \sin 2x;\ \cos 2x;\ \ldots;\ \sin kx;\ \cos kx;\ \ldots\} \tag{8.21}$$

The series with respect to system of trigonometric functions

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{8.22}$$

is called *trigonometric series*. We shall see later that taking the constant term as $\frac{a_0}{2}$ rather that a_0 is a convenience that enables us to make a_0 fit a general result.

Suppose the function $f(x)$ is 2π-periodic i.e. for each $x, x + 2\pi \in X$

$$f(x + 2\pi) = f(x)$$

which means that the values of the function are repeated at interval 2π in its domain. For this 2π-periodic function we find coefficients of trigonometric series (8.22)

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx \tag{8.23}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx \quad k = 1,\ 2,\ \ldots \tag{8.24}$$

and

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx \quad k = 1,\ 2,\ \ldots \tag{8.25}$$

The coefficients a_0, a_k and b_k defined by (8.23), (8.24) and (8.25), respectively, are called the *Fourier coefficients* of the function $f(x)$ and the trigonometric series with these coefficients is called the *Fourier series* of the function $f(x)$.

We have got the formulas to compute the Fourier coefficients. But if we compute the Fourier coefficients by the formulas (8.23), (8.24) and (8.25) and write the Fourier series expansion

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

we don't know whether this expansion converges and if it converges, converges it to $f(x)$ or to some other value. For now we are just saying that associated with the function $f(x)$ on $[-\pi; \pi]$ is a certain series called Fourier series. Therefore we write

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \tag{8.26}$$

The equality sign $=$ can be written instead of $\sim$ only if we have proved the convergence of the Fourier series to the function $f(x)$.

Example 1. Find the Fourier coefficients and Fourier series of the square-wave function defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad -\pi < x \leq 0 \\ 1 & \text{if} \quad 0 < x \leq \pi \end{cases} \qquad \text{and} \quad f(x + 2\pi) = f(x)$$

So $f(x)$ is periodic with period 2π. Using the formulas (8.23), (8.24) and (8.25), we find the Fourier coefficients

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{0}^{\pi} dx = \frac{1}{\pi} \cdot \pi = 1$$

$$a_k = \frac{1}{\pi} \int_{0}^{\pi} \cos kx dx = \frac{1}{k\pi} \sin kx \Big|_0^{\pi} = 0$$

and

$$b_k = \frac{1}{\pi} \int_{0}^{\pi} \sin kx dx = -\frac{1}{k\pi} \cos kx \Big|_0^{\pi} = -\frac{1}{k\pi}((-1)^k - 1) = \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{2}{k\pi} & \text{if } k \text{ is odd} \end{cases}$$

Thus, $a_k = 0$ and and $b_{2k} = 0$ for every $k = 1, 2, \ldots$. Fourier series of square-wave function is

$$f(x) \sim \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \ldots$$

or

$$f(x) \sim \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi}\sin(2k+1)x$$

The following theorem gives a sufficient condition for convergence of the Fourier series.

Theorem (Dirichlet's theorem). If $f(x)$ is a bounded 2π-periodic function which in any one period has at most a finite number of local maxima and minima and a finite number of points of jump discontinuity, then the Fourier series of $f(x)$ converges to $f(x)$ at all points where $f(x)$ is continuous and converges to the average of the right- and left-hand limits of $f(x)$ at each point where $f(x)$ is discontinuous.

The square-wave function has on half-open interval $(-\pi; \pi]$ one local maximum equal to 1 and one local minimum equal to 0 and two points of jump discontinuity 0 an π. Hence, at any point in the open intervals $(-\pi; 0)$ and $(0; \pi)$ Fourier series converges to $f(x)$. The left-hand limit at 0 is $f(0-) = \lim\limits_{x\to 0-} f(x) = 0$ and the right-hand limit at 0 is $f(0+) = \lim\limits_{x\to 0+} f(x) = 1$ and the average of these one-sided limits is $\frac{0+1}{2} = \frac{1}{2}$. The left-hand limit at π is $f(\pi-) = \lim\limits_{x\to\pi-} f(x) = 1$ and the right-hand limit at π is $f(\pi+) = \lim\limits_{x\to\pi+} f(x) = 0$ and the average of one-sided limits is $\frac{1+0}{2} = \frac{1}{2}$.

Thus, at the points of discontinuity the Fourier series of the square-wave function converges to $\frac{1}{2}$. Since $\sin((2k+1)\cdot 0) = 0$ and $\sin((2k+1)\pi) = 0$ for any integer k, then the direct computation also gives

$$\frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi}\sin((2k+1)0) = \frac{1}{2}$$

and

$$\frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi}\sin((2k+1)\pi) = \frac{1}{2}$$