1 Functions on several variables

To any ordered pair of real numbers (x, y) there is related one point in xy-plane. To any point in xy-plane there are related the coordinates of this point, that means the ordered pair of real numbers. It is said that between ordered pairs of real numbers and the points on xy-plane there is *one-to-one correspondence.*

The subset of the points of xy-plane is called the domain (region). We shall denote the domains by D. For example the domain

$$D = \{(x, y)|\ x^2 + y^2 \leq 1\}$$

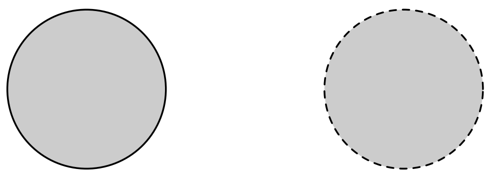
is the unit disk centered at the origin, which contains the circle surrounding this disk.

The curve bounding the domain is called the *boundary line* of this domain and the points on the boundary line are called *boundary points*. The points not laying on the boundary line are called *interior points*.

The domain containing all of its boundary points (that means the whole boundary line) is called *closed*.

The domain containing none of its boundary points is called *open* (if it contains some but not all of its boundary points, then it is neither open or closed).

If the domain contains its boundary line or a part of its boundary line, we sketch this line (part of the line) by the continuous line. If the domain does not contain its boundary line or a part of its boundary line, we sketch this line (part of the line) by the dashed line.



Joonis 1.1: Closed and open disk

Any open disk centered at the given point is called the *neighborhood* of this point. If $\delta > 0$ is whatever real number, then the δ-neighborhood of the point(x_0, y_0) is the open disk (without center)

$$U_\delta(x_0, y_0) = \{(x, y)|\ 0 < (x - x_0)^2 + (y - y_0)^2 < \delta^2\}$$

There exists the one-to-one correspondence between the triplets of real numbers (x, y, z) and the points in space. The subset of the (x, y, z)-space is called the *spatial region*.

The spatial region is separated from the rest of the space by a surface, which is called the *boundary surface*. The points on the boundary surface are called the boundary points and the points not laying on the boundary surface are called interior points.

The region is called closed, if it contains all of its boundary points and the region is called open, if it contains none of its boundary points.

Thus, the closed region is the region with the surface surrounding the region and the open region is the region without the surface surrounding the region.

The δ-neighborhood of the spatial point (x_0, y_0, z_0) is the open ball

$$U_\delta(x_0, y_0, z_0) = \{(x, y, z)|\ 0 < (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \delta^2\}$$

that means the open ball centered at (x_0, y_0, z_0) with radius δ. This open ball does not contain the sphere surrounding the ball and does not contain the center (x_0, y_0, z_0).

1.1 Functions of two variables

Let D be some domain in the xy-plane (included the whole plane). A function of two variables is a function whose inputs are points $(x; y)$ in the xy-plane and whose outputs real numbers.

Definition 1. If to each point $(x; y) \in D$ there is related one certain value of the variable z, then z is called the function of two variables x and y and denoted

$$z = f(x, y)$$

The function of two variables can be also denoted by $z = g(x, y)$, $z = F(x, y)$, $z = \varphi(x, y)$ or $z = z(x, y)$.

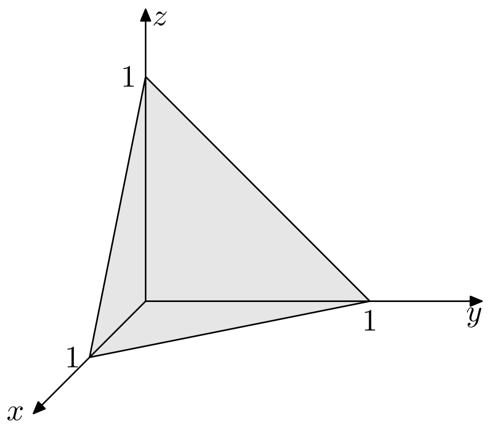
The variables x and y are the *independent variables* and z is the function or the *dependent variable*.

Whenever a quantity depend on two others we have a function of two variables. The area on the rectangle of length x and width y is $S = xy$. The number of items n which can be sold is the function of the price p and the advertising budget a that is $n = f(p, a)$. The force of the suns gravity F depends on an object mass m and the distance d: $F = F(m, d)$.

Further we shall consider the functions given implicitly. In those cases to each point $(x; y) \in D$ there can be related two or more values of the variable z. We talk about the two-valued functions, three-valued functions, etc.

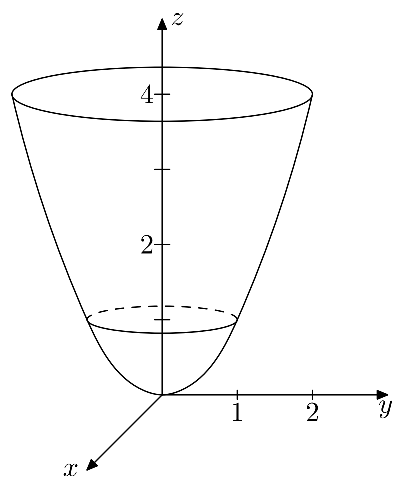
In the graph of the function of two variables $z = f(x, y)$ is the spatial point with coordinates $(x, y, f(x, y))$. The set of all those point is the surface in space. Hence, the graph of the function of two variables $z = f(x, y)$ is the surface in x, y, z-coordinates.

Example 1. The graph of the function $z = 1 - x - y$ is the plane.



Joonis 1.2: The graph of the function $z = 1 - x - y$ in the I octant

Example 2. The graph of the function $z = x^2 + y^2$ is the paraboloid of revolution created by the rotation of the parabola $z = y^2$ around z-axis.

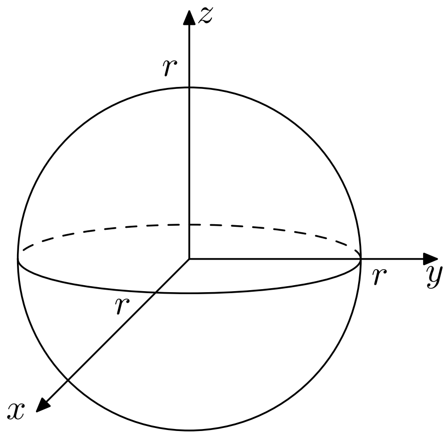


Joonis 1.3: Paraboloid of revolution

The next surface is the graph of the function of two variables given implicitly.

Example 3. The graph of the function $x^2 + y^2 + z^2 = r^2$ given implicitly is the sphere with radius r centered at the origin.

Solving this equation for the variable z, we obtain two one-valued functions of two variables $z = \sqrt{r^2 - x^2 - y^2}$ and $z = -\sqrt{r^2 - x^2 - y^2}$. The graph of the first function is the upper side and second function the lower side of the sphere.



Joonis 1.4: The sphere

Definition 2. The domain of the function of two variables $z = f(x, y)$ is the set of ordered pairs (x, y) (the points of the plane) for which by the given rule it is possible to evaluate the value of the function.

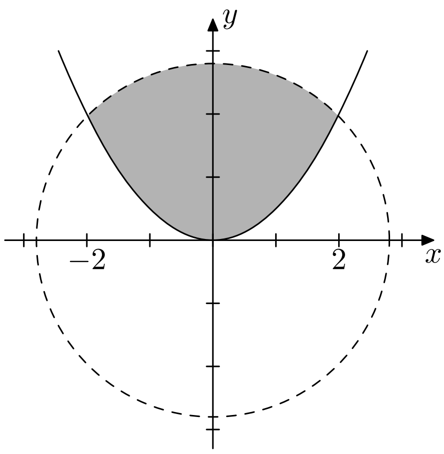
Example 4. Let us find the domain of the function $z = \ln(8 - x^2 - y^2) + \sqrt{2y - x^2}$ and sketch it in the coordinate plane.

The function is defined if there hold two conditions

$$\begin{cases} 8 - x^2 - y^2 > 0 \\ 2y - x^2 \geq 0. \end{cases}$$

The first condition yields $x^2 + y^2 < 8$ and the second $y \geq \frac{x^2}{2}$. The first condition holds for the points in xy-plane, which belong to the disk centered at the origin and with radius $2\sqrt{2}$. The is no equality to 8, therefore the circle surrounding the disk does no belong to the set and we sketch the circle with dashed line.

The second condition holds for the points in xy-plane, which are above the parabola $y = \frac{x^2}{2}$. This condition contains the equality, consequently the parabola belongs to the set and we sketch it with continuous line.

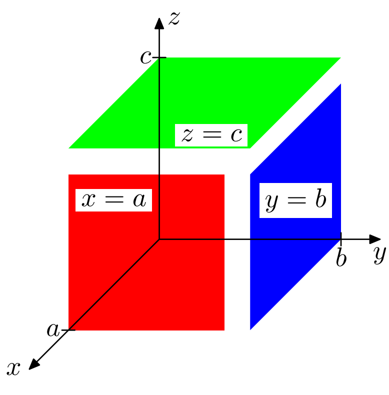


Joonis 1.5: The domain of the function $z = \ln(8 - x^2 - y^2) + \sqrt{2y - x^2}$

1.2 Plane sections and level curves of graph of function of two variables

To get an idea how does the graph of the function of two variables looks like it is useful to sketch plane sections of this surface by the planes which are perpendicular to one of the coordinate axis (i.e. parallel to one of the coordinate planes). The equation of the yz-plane is $x = 0$, the equation of the xz-plane is $y = 0$ and the equation of the xy-plane is $z = 0$.

The plane $x = a$ is perpendicular to x-axes, i.e. parallel to yz-plane; the plane $y = b$ is perpendicular to y-axes i.e. parallel to xz-plane; the plane $z = c$ is perpendicular to z-axes i.e. parallel to xy-plane.



Joonis 1.6: The planes $x = a$, $y = b$ and $z = c$

The intersections of the surface $z = f(x, y)$ with the planes $x = a$ are the curves

$$\begin{cases} z = f(x, y) \\ x = a, \end{cases}$$

The intersections of the surface $z = f(x; y)$ with the planes $y = b$ are the curves

$$\begin{cases} z = f(x, y) \\ y = b \end{cases}$$

The intersections of the surface $z = f(x; y)$ with the planes $z = c$ are the

$$\begin{cases} z = f(x, y) \\ z = c, \end{cases}$$

The projection of the resulting curve onto the xy-plane is called the *level curve*. A collection of level curves of a surface is called a *contour map*.

Example 1. Let us sketch the surface $x^2 + y^2 - z^2 = 0$, using the intersections with the planes $z = 0$, $z = \pm 1$, $z = \pm 2$ and $x = 0$. First five are the horizontal curves and the sixth is the intersection with the yz-plane.

The intersection of this surface with xy-plane $z = 0$ is actually one point determined by the equations $x^2 + y^2 = 0$, $z = 0$, which is the origin.

The intersection of this surface with the horizontal plane $z = 1$ is the circle $x^2 + y^2 = 1$, $z = 1$, the unit circle on the plane $z = 1$ centered at $(0; 0; 1)$.

The intersection of this surface with the horizontal plane $z = -1$ is the unit circle $x^2 + y^2 = 1$ again but centered at $(0; 0; -1)$.

The intersection of this surface with the horizontal plane $z = 2$ is the circle $x^2 + y^2 = 4$ centered at $(0; 0; 2)$ with radius 2.

The intersection of this surface with the horizontal plane $z = -2$ is the circle $x^2 + y^2 = 4$ centered at $(0; 0; -2)$ with radius 2.

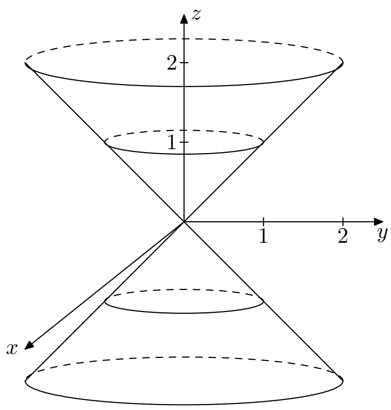
The intersection of this surface with the vertical plane $x = 0$ is determined by $z^2 = y^2$, $x = 0$, that is two perpendicular lines $z = y$ and $z = -y$ on yz-plane. Adding these two lines to our sketch it turns obvious that the given surface is the cone, whose vertex is at the origin.

If we convert the function $x^2 + y^2 - z^2 = 0$ to the explicit form we obtain two one-valued functions $z = \sqrt{x^2 + y^2}$ and $z = -\sqrt{x^2 + y^2}$. The graph of the first function is the upper part of the cone and the graph of the second function is the lower part of the cone.

Example 2. Let us sketch the surface $z = x^2 - y^2$, using the intersections with the planes $y = 0$, $x = \pm 1$, $x = \pm 0,5$, $x = 0$, $z = 0$ and $z = -0,44$.

In this example we draw the coordinate axes in an unusual way, taking the sheet of paper the xz-plane and directing y-axes backwards.

The intersection with the plane $y = 0$ is the parabola $z = x^2$, $y = 0$.



Joonis 1.7: Intersections of the cone $x^2 + y^2 - z^2 = 0$ by planes $z = \pm 1$, $z = \pm 2$ and $x = 0$

The intersections with the planes $x = \pm 1$ are the parabolas $z = 1 - y^2$, $x = 1$ and $z = 1 - y^2$, $x = -1$.

The intersections with the planes $x = \pm 0,5$ are the parabolas $z = 0,25 - y^2$, $x = 0,5$ and$z = 0,25 - y^2$, $x = -0,5$.

The intersections with the plane $z = 0$ are two perpendicular lines $y = x$ and $y = -x$ on the xy-plane.

The intersection with the plane $z = -0,44$ is the equilateral hyperbola $y^2 - x^2 = 0,44$, whose real axis is the y-axis.

The *level surfaces* of the graph of function of three variables $w = f(x, y, z)$ are the surfaces

$$\begin{cases} w = f(x, y, z) \\ w = c. \end{cases}$$

This system of equations yields the equation $f(x, y, z) = c$, the function of two variables given implicitly, whose graph is a surface in the space.

Example 3. The level surfaces of the function $w = x^2 + y^2 + z^2$ are $x^2 + y^2 + z^2 = c$ provided $c > 0$. Those surfaces are the spheres centered at the origin with radius $\sqrt{c}$.

1.3 Increment of function of several variables

Let us fix one point $P(x, y)$ in the domain of the function $z = f(x, y)$. Changing the variable x by Δx and y by Δy, we obtain a point $Q(x+\Delta x, y+\Delta y)$. Assuming that the increments of the independent variables Δx and Δy are sufficiently small, that is Q is also in the domain of the function, we define the *total increment* of the function

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) \tag{1.1}$$

Assuming that y is a constant or $\Delta y = 0$, we have the increment of the function with respect to variable x.

$$\Delta_x z = f(x + \Delta x, y) - f(x, y) \tag{1.2}$$

Assuming that x is a constant or $\Delta x = 0$, we have the increment of the function with respect to variable y.

$$\Delta_y z = f(x, y + \Delta y) - f(x, y) \tag{1.3}$$

One might guess that $\Delta z = \Delta_x z + \Delta_y z$ but as the following example proves, this is not true.

Example 1. For the function $z = xy$ let us find Δz and $\Delta_x z + \Delta_y z$ if $x = 2$, $y = 3$, $\Delta x = 0,2$ and $\Delta y = 0,1$.

First $\Delta_x z = (x + \Delta x)y - xy = y\Delta x = 3 \cdot 0,2 = 0,6$,

second $\Delta_y z = x(y + \Delta y) - xy = x\Delta y = 2 \cdot 0,1 = 0,2$. Thus, $\Delta_x z + \Delta_y z = 0,8$.

The total increment of the function $\Delta z = (x + \Delta x)(y + \Delta y) - xy = x\Delta y + y\Delta x + \Delta x\Delta y = 2 \cdot 0,1 + 3 \cdot 0,2 + 0,2 \cdot 0,1 = 0,82$.

The total increment of the function of three variables $w = f(x, y, z)$ is defined as

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

If y and z are constants, we define

$$\Delta_x w = f(x + \Delta x, y, z) - f(x, y, z)$$

if x and z are constants, we define

$$\Delta_y w = f(x, y + \Delta y, z) - f(x, y, z)$$

and if x and y are constants, we define

$$\Delta_z w = f(x, y, z + \Delta z) - f(x, y, z)$$

1.4 Limit and continuity of functions of two variables

Suppose $P_0(x_0, y_0)$ is a fixed point in the domain of the function $z = f(x, y)$ and $P(x, y)$ is a moving point that approaches P_0. We shall write $(x, y) \to (x_0, y_0)$ or $x \to x_0$, $y \to y_0$.

To find the limit of a function of one variable, we only needed to test the approach from the left and the approach from the right. If both approaches gave the same result, the function had a limit. To find the limit of a function of two variables however, we must show that the limit is the same no matter from which direction we approach (x_0, y_0)

The moving point P can approach the fixed point P_0 along whatever path: along the straight line, broken line, the arc of parabola etc. Independently of the path, the moving point P reaches to any neighborhood of $U_\delta(x_0, y_0)$ for arbitrary small $\delta > 0$.

Definition 1. The real number L is called the limit of the function $f(x, y)$ in limiting process $(x, y) \to (x_0, y_0)$, if $\forall\, \varepsilon > 0$ there exists the neighborhood $U_\delta(x_0, y_0)$ such that $|f(x, y) - L| < \varepsilon$ whenever $(x, y) \in U_\delta(x_0, y_0)$

In other words, L is the limit of the function $f(x, y)$ as $(x, y) \to (x_0, y_0)$, if the value of the function $f(x, y)$ can be made as close as desired to L by taking $P(x, y)$ in the neighborhood of $P_0(x_0, y_0)$ small enough.

This is denoted

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

Example 1. Find the limit $\lim\limits_{(x,y)\to(0;0)} \dfrac{xy}{x^2 + y^2}$.

Let $(x, y) \to (0; 0)$ along the line $y = kx$. Then

$$\frac{xy}{x^2 + y^2} = \frac{x \cdot kx}{x^2 + k^2x^2} = \frac{k \cdot x^2}{x^2(1 + k^2)} = \frac{k}{1 + k^2}$$

This shows that the result depends on the choice of the slope of the line k. Therefore, the limit does not exist.

Often it is useful to convert the limit into polar coordinates, taking $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$. Then $x^2 + y^2 = \rho^2$ and the limiting process $(x, y) \to (0; 0)$ is equivalent to $\rho \to 0$. In Example 1 we could write

$$\lim_{(x,y)\to(0;0)} \frac{xy}{x^2 + y^2} = \lim_{\rho\to 0} \frac{\rho\cos\varphi\rho\sin\varphi}{\rho^2\cos^2\varphi + \rho^2\sin^2\varphi} = \lim_{\rho\to 0} \frac{\rho^2\cos\varphi\sin\varphi}{\rho^2} = \cos\varphi\sin\varphi$$

The result depends on the polar angle and this proves again that the limit does not exist.

Example 2. Find the limit $\lim\limits_{(x,y)\to(0;0)} \dfrac{\sin(x^2 + y^2)}{x^2 + y^2}$.

Converting this limit into polar coordinates, we have

$$\lim_{(x,y)\to(0;0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim_{\rho\to 0} \frac{\sin(\rho^2\cos^2\varphi + \rho^2\sin^2\varphi)}{\rho^2\cos^2\varphi + \rho^2\sin^2\varphi} = \lim_{\rho\to 0} \frac{\sin\rho^2}{\rho^2} = 1$$

Definition 2. The function $f(x, y)$ is called continuous at the point $P_0(x_0, y_0)$, if

1. $\exists f(x_0, y_0)$

2. $\exists \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$

3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

Definition 3. The function is called continuous in the domain D, if it is continuous at every point of this domain.

Let us denote the fixed point in Definition 2 by (x, y) and the moving point by $(x + \Delta x, y + \Delta y)$. Then $(x + \Delta x, y + \Delta y) \to (x, y)$ if and only if $(\Delta x, \Delta y) \to (0; 0)$. The third condition of continuity can be re-written

$$\lim_{(\Delta x,\Delta y)\to(0;0)} f(x+\Delta x, y+\Delta y) = f(x,y)$$

or

$$\lim_{(\Delta x,\Delta y)\to(0;0)} [f(x+\Delta x, y+\Delta y) - f(x,y)] = 0 \tag{1.4}$$

In square brackets of the last condition there is the total increment Δz of the function $z = f(x, y)$ and the condition of the continuity (1.4) at the point (x, y) is

$$\lim_{(\Delta x,\Delta y)\to(0;0)} \Delta z = 0 \tag{1.5}$$

The equality (1.5) is called the **necessary and sufficient condition of continuity**.