6 Line integral

Line integral is an integral where the function to be integrated is evaluated along a curve. The terms path integral, curve integral, and curvilinear integral are also used.

6.1 Line integral with respect to arc length

Suppose that on the plane curve AB there is defined a function of two variables $f(x, y)$, i.e. to any point (x, y) of this curve there is related the value $f(x, y)$. Let

$$A = P_0,\ P_1,\ P_2,\ \ldots, P_{k-1},\ P_k,\ \ldots,\ P_n = B$$

the random partition of the curve AB into subarcs $\widehat{P_{k-1}P_k}$, $k = 1,\ 2,\ \ldots,\ n$. From every subarc we pick a random point $Q_k(\xi_k, \eta_k) \in \widehat{P_{k-1}P_k}$.

Denote by Δs_k the length of the subarc $\widehat{P_{k-1}P_k}$. Now we multiply the value at the point chosen by the length of subarc $f(Q_k)\Delta s_k$, where $k = 1,\ 2,\ \ldots,\ n$. Adding all those products, we get the sum

$$s_n = \sum_{k=1}^{n} f(Q_k)\Delta s_k \tag{6.1}$$

which is called the *integral sum* of the function $f(x, y)$ *over the curve* AB.

We have the random partition of the curve AB. Therefore, the lengths Δs_k of subarcs $\widehat{P_{k-1}P_k}$ are different. Denote by λ the greatest length of subarcs, i.e.

$$\lambda = \max_{1 \le k \le n} \Delta s_k$$

Definition. If there exists the limit

$$\lim_{\lambda \to 0} s_n$$

and this limit does not depend on the partition of AB and does not depend on the choice of the points Q_k on the subarcs, then this limit is called the *line integral with respect to arc length* and denoted by

$$\int\limits_{AB} f(x, y) ds$$

Thus, by the definition

$$\int\limits_{AB} f(x,y)ds = \lim_{\lambda\to 0}\sum_{k=1}^{n} f(Q_k)\Delta s_k$$

Line integral with respect to arc length is also referred as *line integral of a scalar field* because $f(x,y)$ defines a scalar field on the curve AB.

Suppose the curve AB is the piece of wire. If the function $\rho(x,y)\geq 0$ represents the density (mass per unit length) for wire AB, then the product $\rho(Q_k)\Delta s_k$ is the approximate mass of subarc Δs_k and the integral sum

$$\sum_{k=1}^{n}\rho(Q_k)\Delta s_k$$

is the approximate mass of the wire AB. For shorter subarc the value $\rho(Q_k)$ represents the variable density $\rho(x,y)$ of subarc with greater accuracy. Thus, in this case the limit of the integral sum, i.e. the line integral with respect to arc length gives the mass of the wire AB:

$$m = \int\limits_{AB}\rho(x,y)ds \tag{6.2}$$

The properties on the line integral with respect to arc length can be proved directly, using the definition.

Property 1. The line integral with respect to arc length does not depend on the direction the curve AB has been traversed:

$$\int\limits_{AB} f(x,y)ds = \int\limits_{BA} f(x,y)ds$$

Property 2. (Additivity property) If C is some point on the curve AB, then

$$\int\limits_{AB} f(x,y)ds = \int\limits_{AC} f(x,y)ds + \int\limits_{CB} f(x,y)ds$$

Property 3.

$$\int\limits_{AB}[f(x,y)\pm g(x,y)]ds = \int\limits_{AB} f(x,y)ds \pm \int\limits_{AB} g(x,y)ds$$

Property 4. If c ic a constant, then

$$\int\limits_{AB} cf(x,y)ds = c\int\limits_{AB} f(x,y)ds$$

Property 5. Taking in the definition of the line integral with respect to arc length $f(x, y) \equiv 1$, we get the integral sum

$$s_n = \sum_{k=1}^{n} \Delta s_k$$

which is the sum of lengths of subarcs. This is the length of arc AB for any partition. Thus, for $f(x, y) \equiv 1$ the line integral gives us the length of arc AB:

$$s_{AB} = \int\limits_{AB} ds$$

Property 5 can be also obtained by taking in (6.2) the density $\rho(x, y) \equiv 1$ because then the mass and the length of the curve are numerically equal.

Any point of the curve AB in the space has three coordinates $Q_k(\xi_k, \eta_k, \zeta_k)$. So, the function defined on the space curve is in general a function of three variables $f(x, y, z)$. Defining the line integral with respect to arc length along the space curve we do everything like we did in the definition for the two-dimensional case:

$$\int\limits_{AB} f(x, y, z) ds = \lim_{\lambda \to 0} \sum_{k=1}^{n} f(Q_k) \Delta s_k \tag{6.3}$$

Of course, five properties of the line integral for three-dimensional case are still valid.

6.2 Evaluation of line integral with respect to arc length

Suppose that the parametric equations of the curve AB in the plane are

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

and the parametric equations of the curve AB in the space are

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t), \end{cases}$$

where at the point A the value of the parameter $t = \alpha$ and at the point B the value of the parameter $t = \beta$.

Definition 1. The plane curve AB is called smooth, if $\dot{x} = \dfrac{dx}{dt}$ and $\dot{y} = \dfrac{dy}{dt}$ are continuous on $[\alpha; \beta]$ and

$$\dot{x}^2 + \dot{y}^2 \neq 0$$

Definition 2. The curve AB in the space is called smooth, if $\dot{x} = \dfrac{dx}{dt}$, $\dot{y} = \dfrac{dy}{dt}$ and $\dot{z} = \dfrac{dz}{dt}$ are continuous on $[\alpha; \beta]$ and

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \neq 0$$

Intuitively, a smooth curve is one that does not have sharp corners.

Theorem 1. If the function $f(x, y)$ is continuous on the smooth curve AB, then

$$\int\limits_{AB} f(x, y)ds = \int\limits_{\alpha}^{\beta} f[x(t), y(t)]\sqrt{\dot{x}^2 + \dot{y}^2}dt \qquad (6.4)$$

Theorem 2. If the function $f(x, y, z)$ is continuous on the smooth curve AB, then

$$\int\limits_{AB} f(x, y, z)ds = \int\limits_{\alpha}^{\beta} f[x(t), y(t), z(t)]\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}dt \qquad (6.5)$$

If $\mathbf{r}(t) = (x(t), y(t), z(t))$ is the position vector of a point on the curve, then the square root in the formula (6.5) is the length of $\dot{\mathbf{r}}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$ i.e $|\dot{\mathbf{r}}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$. The formula (6.5) can be re-written as

$$\int\limits_{AB} f(x, y, z)ds = \int\limits_{\alpha}^{\beta} f[x(t), y(t), z(t)]|\dot{\mathbf{r}}(t)|dt$$

Suppose the curve AB is a graph of the function $y = \varphi(x)$ given explicitly, at the point A $x = a$ and at B $x = b$. The curve is smooth, if there exists $\varphi'(x)$ on the interval $[a; b]$.

Theorem 3. If the function $f(x, y)$ is continuous on the smooth curve AB, then

$$\int\limits_{AB} f(x, y)ds = \int\limits_{a}^{b} f[x, \varphi(x)]\sqrt{1 + y'^2}dx \qquad (6.6)$$

This theorem is the direct conclusion of Theorem 1 because treating the variable x as the parameter, we have $\dot{x} = 1$ and $\dot{y} = \dfrac{dy}{dx} = y'$.

Example 1. Compute the line integral $\int\limits_{AB} \dfrac{ds}{x-y}$, where AB is the segment of the line $y = 2x - 3$ between coordinate axes.

The line is the graph of the function given explicitly. Therefore, we use for the computation the formula (6.6).

At the intersection point by y axis $x = 0$ and at the intersection point by x axis $y = 0$, i.e. $x = \dfrac{3}{2}$. To apply the formula, we find $y = 2$ and $1 + y'^2 = 5$. Thus,

$$\begin{aligned}\int\limits_{AB} \frac{ds}{x-y} &= \int\limits_0^{\frac{3}{2}} \frac{\sqrt{5}dx}{x-(2x-3)} = \sqrt{5}\int\limits_0^{\frac{3}{2}} \frac{dx}{3-x} = -\sqrt{5}\int\limits_0^{\frac{3}{2}} \frac{d(3-x)}{3-x} \\ &= -\sqrt{5}\ln|3-x|\Big|_0^{\frac{3}{2}} = -\sqrt{5}\left(\ln\frac{3}{2} - \ln 3\right) = -\sqrt{5}\ln\frac{1}{2} = \sqrt{5}\ln 2\end{aligned}$$

Example 2. Compute the line integral $\int\limits_{AB} \sqrt{y}ds$, where AB is the first arc of cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

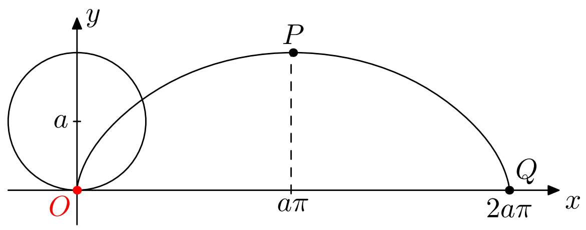


Figure 6.1: cycloid

For the first arc of cycloid $0 \le t \le 2\pi$. To apply the formula (6.4), we find $\dot{x} = a(1 - \cos t)$, $\dot{y} = a\sin t$ and

$$\dot{x}^2+\dot{y}^2 = a^2(1-\cos t)^2+a^2\sin^2 t = a^2(1-2\cos t+\cos^2 t+\sin^2 t) = 2a^2(1-\cos t)$$

By the formula (6.4)

$$\int\limits_{AB} \sqrt{y}ds = \int\limits_0^{2\pi} \sqrt{a(1-\cos t)}\sqrt{2a^2(1-\cos t)}dt =$$

$$a\sqrt{2a}\int\limits_0^{2\pi}(1-\cos t)dt = a\sqrt{2a}(t-\sin t)\Big|_0^{2\pi} = 2\pi a\sqrt{2a}$$

Example 3. Compute the line integral $\int\limits_{AB}(2z-\sqrt{x^2+y^2})ds$, where AB is the first turn of conical helix $x = t\cos t$, $y = t\sin t$, $z = t$.

For the first turn of conical helix $0 \leq t \leq 2\pi$. Find $\dot{x} = \cos t - t\sin t$, $\dot{y} = \sin t + t\cos t$, $\dot{z} = 1$ and

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1 =$$
$$\cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + 1 = 2 + t^2$$

By the formula (6.5) we obtain

$$\int\limits_{AB}(2z-\sqrt{x^2+y^2})ds = \int\limits_0^{2\pi}(2t-\sqrt{t^2\cos^2 t + t^2\sin^2 t})\sqrt{2+t^2}dt =$$

$$\int\limits_0^{2\pi}(2t-t)\sqrt{2+t^2}dt = \int\limits_0^{2\pi} t\sqrt{2+t^2}dt = \frac{1}{2}\int\limits_0^{2\pi}\sqrt{2+t^2}d(2+t^2) =$$

$$\frac{1}{2}\frac{(2+t^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^{2\pi} = \frac{(2+t^2)^{\frac{3}{2}}}{3}\Bigg|_0^{2\pi} = \frac{(2+4\pi^2)\sqrt{2+4\pi^2}-2\sqrt{2}}{3}$$

6.3 Line integral with respect to coordinates

In the first subsection we defined the line integral for the scalar field. Now we are going to define the line integral for the vector field. First we consider the two-dimensional case. Let AB be the curve in the plane and $\overrightarrow{F} = (X(x,y); Y(x,y))$ a force vector. Suppose that the force is applied to an object to move it along the curve AB. The goal in to find the work done by this force. To do it, we first divide the curve AB with the points

$$A = P_0,\ P_1,\ \ldots,\ P_{k-1},\ P_k,\ \ldots,\ P_n = B$$

into subarcs $\widehat{P_{k-1}P_k}$, where $k = 1,\ 2,\ \ldots,\ n$ and approximate any subarc $\widehat{P_{k-1}P_k}$ to the vector $\overrightarrow{P_{k-1}P_k}$.

Denote the coordinates of the kth partition point P_k by x_k and y_k, i.e. $P_k(x_k; y_k)$ and the coordinates of the vector $\overrightarrow{P_{k-1}P_k}$ by

$$\Delta x_k = x_k - x_{k-1}$$

and

$$\Delta y_x = y_k - y_{k-1}$$

that is

$$\overrightarrow{P_{k-1}P_k} = (\Delta x_k; \Delta y_k)$$

Let Δs_k be the magnitude of the vector $\overrightarrow{P_{k-1}P_k}$:

$$\Delta s_k = \sqrt{\Delta x_k^2 + \Delta y_k^2}$$

and λ the greatest of all those magnitudes

$$\lambda = \max_{1 \le k \le n} \Delta s_k$$

Next we choose a random point $Q_k(\xi_k; \eta_k)$ on any subarc $\widehat{P_{k-1}P_k}$ and substitute on this subarc the variable force vector by the constant force vector

$$\overrightarrow{F_k} = (X(\xi_k, \eta_k); Y(\xi_k, \eta_k))$$

Recall that if a constant force $\overrightarrow{F_k}$ is applied to an object to move it along a straight line from the point P_{k-1} to the point P_k, then the amount of work done A_k is the scalar product of the force vector and the vector $\overrightarrow{P_{k-1}P_k}$:

$$A_k = \overrightarrow{F_k} \cdot \overrightarrow{P_{k-1}P_k} = X(\xi_k, \eta_k)\Delta x_k + Y(\xi_k, \eta_k)\Delta y_k$$

The total work done by the force vector $\overrightarrow{F}$, moving an object from the point A to the point B along the curve is approximately

$$\sum_{k=1}^{n} [X(\xi_k, \eta_k)\Delta x_k + Y(\xi_k, \eta_k)\Delta y_k]. \tag{6.7}$$

Approximately because we have approximated the subarc $\widehat{P_{k-1}P_k}$ to the vector $\overrightarrow{P_{k-1}P_k}$ and the variable force vector $\overrightarrow{F} = (X(x, y); Y(x, y))$ to the constant vector $\overrightarrow{F_k} = (X(\xi_k, \eta_k); Y(\xi_k, \eta_k))$.

Obviously, taking more partition points, the subarcs get shorter and the vectors $\overrightarrow{P_{k-1}P_k}$ represent the subarcs $\widehat{P_{k-1}P_k}$ with greater accuracy. As well, the constant vector $\overrightarrow{F_k} = (X(\xi_k, \eta_k); Y(\xi_k, \eta_k))$ represents the variable vector $\overrightarrow{F} = (X(x, y); Y(x, y))$ on $\overset{\frown}{P_{k-1}P_k}$ with greater accuracy.

Definition. If the sum (6.7) has the limit as $\max \Delta s_k \to 0$ and this limit does not depend on the partition of the curve AB and does not depend on the choice of points Q_k on subarcs, then this limit is called the *line integral with respect to coordinates* and denoted

$$\int_{AB} X(x, y)dx + Y(x, y)dy$$

Thus, by the definition

$$\int_{AB} X(x, y)dx + Y(x, y)dy = \lim_{\lambda \to 0} \sum_{k=1}^{n} [X(\xi_k, \eta_k)\Delta x_k + Y(\xi_k, \eta_k)\Delta y_k] \quad (6.8)$$

If AB is a curve in the space, then

$$\overrightarrow{P_{k-1}P_k} = (\Delta x_k; \Delta y_k; \Delta z_k)$$

and the magnitude of this vector

$$\Delta s_k = \sqrt{\Delta x_k^2 + \Delta y_k^2 + \Delta z_k^2}$$

Also the force vector has three coordinates

$$\overrightarrow{F} = (X(x, y, z); Y(x, y, z)); Z(x, y, z))$$

The line integral with respect to coordinates is defined as the limit

$$\int_{AB} X(x, y, z)dx + Y(x, y, z)dy + Z(x, y, z)dz$$

$$= \lim_{\lambda \to 0} \sum_{k=1}^{n} [X(\xi_k, \eta_k, \zeta_k)\Delta x_k + Y(\xi_k, \eta_k, \zeta_k)\Delta y_k + Z(\xi_k, \eta_k, \zeta_k)\Delta z_k]$$

We consider the properties of the line integral with respect to coordinates for the curve in the plane. All of this discussion generalizes to space curves in a straightforward manner.

Property 1. If C is a random point on the curve AB, then

$$\int_{AB} X(x, y)dx + Y(x, y)dy = \int_{AC} X(x, y)dx + Y(x, y)dy + \int_{CB} X(x, y)dx + Y(x, y)dy \quad (6.9)$$

Property 2. If the curve is traced in reverse (that is, from the terminal point to the initial point), then the sign of the line integral is reversed as well:

$$\int\limits_{BA} X(x,y)dx + Y(x,y)dy = -\int\limits_{AB} X(x,y)dx + Y(x,y)dy \tag{6.10}$$

6.4 Evaluation of line integral with respect to coordinates

Suppose that AB is a smooth curve in the plane

$$x = x(t),\ \ y = y(t)$$

and the functions $X(x,y)$ and $Y(x,y)$ are continuous on AB. Let at the point A the parameter $t = \alpha$ and at the point B $t = \beta$.

Theorem 1. If the functions $X(x,y)$ and $Y(x,y)$ are continuous on the smooth curve AB, then

$$\int\limits_{AB} X(x,y)dx + Y(x,y)dy = \int\limits_{\alpha}^{\beta}[X(x(t),y(t))\dot{x} + Y(x(t),y(t))\dot{y}]dt \tag{6.11}$$

In three dimensional case there holds the similar theorem. Suppose that on the line AB

$$x = x(t),\ \ y = y(t),\ \ z = z(t)$$

there is defined a vector function $\overrightarrow{F}(x,y,z) = X(x,y,z), Y(x,y,z), Z(x,y,z)$. Suppose again that at the point A the parameter $t = \alpha$ and at the point B $t = \beta$.

Theorem 2. If the functions $X(x,y,z)$, $Y(x,y,z)$ and $Z(x,y,z)$ are continuous on the smooth curve AB, then

$$\int\limits_{AB} X(x,y,z)dx + Y(x,y,z)dy + Z(x,y,z)dz$$
$$= \int\limits_{\alpha}^{\beta}[X(x(t),y(t),z(t))\dot{x} + Y(x(t),y(t),z(t))\dot{y} + Z(x(t),y(t),z(t))\dot{z}]dt \tag{6.12}$$

Conclusion. Suppose the plane curve AB is the graph of the function $y = y(x)$ given explicitly and at the point A $x = a$ and at B $x = b$. Treating the variable x as a parameter, we obtain $\dot{x} = 1$, $\dot{y} = y'$ and by the formula (6.11)

$$\int\limits_{AB} X(x,y)dx + Y(x,y)dy = \int\limits_a^b [X(x,y(x)) + Y(x,y(x))y']dx \tag{6.13}$$

Remark. Sometimes (especially for vertical lines) it is necessary to consider y as the independent variable and x as the function $x = x(y)$. Changing the roles of the variables x and y, we get

$$\int\limits_{AB} X(x,y)dx + Y(x,y)dy = \int\limits_a^b [X(x(y),y)x' + Y(x(y),y)]dy \tag{6.14}$$

A curve L is called *closed* if its initial and final points are the same point. For example a circle is a closed curve. A curve L is called *simple* if it doesn't cross itself. A circle is a simple curve while a figure ∞ type curve is not simple. If L is not a smooth curve, but can be broken into a finite number of smooth curves, then we say that L is *piecewise smooth*. The line integral over the piecewise smooth closed simple curve L is often denoted

$$\oint\limits_L X(x,y)dx + Y(x,y)dy$$

The *positive orientation* of the closed curve L is that as we traverse the curve following the positive orientation the region D bounded by L must always be on the left.

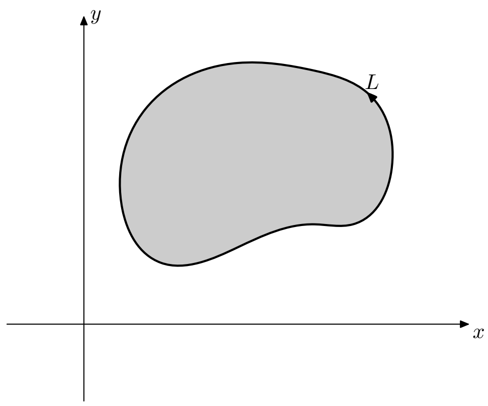


Figure 6.2: Positively oriented closed curve

Example 1. Compute $\int\limits_{AB} x\cos y dx - y\sin x dy$ over the straight line from $A(0;0)$ to $B(\pi;2\pi)$.

The direction vector of the line is $\overrightarrow{AB} = (\pi;2\pi)$ and the parametric equations

$$\begin{aligned} x &= \pi t \\ y &= 2\pi t, \end{aligned}$$

At the point A the parameter $t = 0$ and at the point B $t = 1$. To apply the formula (6.11) we find $\dot{x} = \pi$ and $\dot{y} = 2\pi$. By the formula

$$\begin{aligned} \int\limits_{AB} x\cos y dx - y\sin x dy &= \int\limits_0^1 (\pi t\cos 2\pi t \cdot \pi - 2\pi t\sin \pi t \cdot 2\pi)dt \\ &= \pi^2 \int\limits_0^1 [t(\cos 2\pi t - 4\sin \pi t)]dt = \dots \end{aligned}$$

The integral obtained we integrate by parts, taking

$$u = t, \qquad dv = \cos 2\pi t - 4\sin \pi t$$

Then

$$du = dt, \qquad v = \frac{1}{2\pi}\sin 2\pi t + \frac{4}{\pi}\cos \pi t$$

and

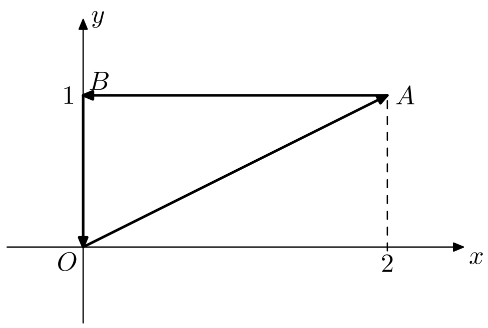
$$\begin{aligned} \dots = \pi^2 &\left[t\left(\frac{1}{2\pi}\sin 2\pi t + \frac{4}{\pi}\cos \pi t\right)\Bigg|_0^1 - \int\limits_0^1 \left(\frac{1}{2\pi}\sin 2\pi t + \frac{4}{\pi}\cos \pi t\right) dt \right] \\ &= \pi^2 \left[-\frac{4}{\pi} + \left(\frac{1}{4\pi^2}\cos 2\pi t - \frac{4}{\pi^2}\sin \pi t\right)\Bigg|_0^1 \right] = -4\pi \end{aligned}$$

Example 2. Compute $\oint\limits_L (x^2 + y)dx + xydy$, where L is the positively oriented triangle OAB with vertices $O(0;0)$, $A(2;1)$ and $B(0;1)$.

The triangle is sketched in Figure 7.3. Notice that the triangle is a simple closed piecewise smooth curve, because it consists of three smooth lines.

By Property 1

$$\oint\limits_L (x^2+y)dx+xydy = \int\limits_{OA} (x^2+y)dx+xydy + \int\limits_{AB} (x^2+y)dx+xydy + \int\limits_{BO} (x^2+y)dx+xydy$$



By Property 2 the direction is important. Compute all three line integrals. The side OA has the equation $y = \dfrac{x}{2}$, $0 \le x \le 2$ and $y' = \dfrac{1}{2}$. By the formula (6.13)

$$\int\limits_{OA} (x^2 + y)dx + xydy = \int\limits_0^2 \left(x^2 + \frac{x}{2} + x \cdot \frac{x}{2} \cdot \frac{1}{2} \right) dx = \int\limits_0^2 \left(\frac{5x^2}{4} + \frac{x}{2} \right) dx$$

The side AB has the equation $y = 1$, hence, $y' = 0$. At the initial point A $x = 2$ and at the end point B $x = 0$. Thus, by (6.13)

$$\int\limits_{AB} (x^2 + y)dx + xydy = \int\limits_2^0 (x^2 + 1 + x \cdot 1 \cdot 0)dx = \int\limits_2^0 (x^2 + 1)dx$$

The third side BO of the triangle is the vertical line $x = 0$, hence, $x' = 0$. At the point B $\ y = 1$ and at the point O $\ y = 0$. To compute the third line integral we use the formula (6.14)

$$\int\limits_{BO} (x^2 + y)dx + xydy = \int\limits_1^0 [(0 + y) \cdot 0 + 0 \cdot y]dy = 0$$

Therefore,

$$\oint\limits_L (x^2 + y)dx + xydy = \int\limits_0^2 \left(\frac{5x^2}{4} + \frac{x}{2} \right) dx + \int\limits_2^0 (x^2 + 1)dx$$

Changing the limits in the last integral gives

$$\oint\limits_L (x^2+y)dx + xydy = \int\limits_0^2 \left(\frac{5x^2}{4} + \frac{x}{2} - x^2 - 1\right) dx$$

$$= \int\limits_0^2 \left(\frac{x^2}{4} + \frac{x}{2} - 1\right) dx = \left(\frac{x^3}{12} + \frac{x^2}{4} - x\right)\Big|_0^2 = \frac{2}{3} + 1 - 2 = -\frac{1}{3}$$

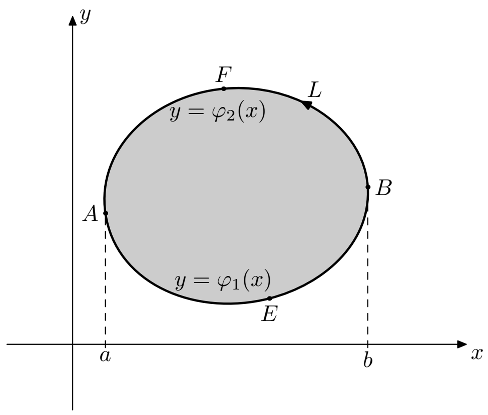
We shall return to the last example once more.

6.5 Green's formula

In this subsection we are going to investigate the relationship between certain kinds of line integrals (on closed curves) and double integrals. Suppose the functions $X(x,y)$ and $Y(x,y)$ are defined on the simple closed curve L and in the region D enclosed by this curve.

Theorem (Green's formula). If the functions $X(x,y)$ and $Y(x,y)$ are continuous on the closed simple piecewise smooth curve L, the partial derivatives $\dfrac{\partial Y}{\partial x}$ and $\dfrac{\partial X}{\partial y}$ are continuous in the regular region D and L is positively oriented, then

$$\oint\limits_L X(x,y)dx + Y(x,y)dy = \iint\limits_D \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) dxdy \tag{6.15}$$



Example. Let us compute the line integral

$$\oint\limits_L (x^2+y)dx + xydy$$

given in Example 2 of the previous subsection once more, using the Green's formula.

Here $X(x, y) = x^2 + y$ and $Y(x, y) = xy$. To apply the Green's formula (6.15) we find $\dfrac{\partial Y}{\partial x} = y$ and $\dfrac{\partial X}{\partial y} = 1$. Let D be the region bounded by L. By the formula (6.15)

$$\oint_L (x^2 + y)dx + xydy = \iint_D (y - 1)dxdy$$

Using Figure 7.3, we determine the limits of integration $0 \le x \le 2$ and $\dfrac{x}{2} \le y \le 1$. Hence,

$$\oint_L (x^2 + y)dx + xydy = \int_0^2 dx \int_{\frac{x}{2}}^1 (y - 1)dy$$

Find the inside integral

$$\int_{\frac{x}{2}}^1 (y - 1)dy = \int_{\frac{x}{2}}^1 (y - 1)d(y - 1) = \left.\frac{(y - 1)^2}{2}\right|_{\frac{x}{2}}^1 = -\frac{\left(\frac{x}{2} - 1\right)^2}{2} = -\frac{(x - 2)^2}{8}$$

and the outside integral

$$\int_0^2 \left[-\frac{(x - 2)^2}{8}\right] dx = -\frac{1}{8}\int_0^2 (x-2)^2 d(x-2) = -\frac{1}{8}\left.\frac{(x - 2)^3}{3}\right|_0^2 = \frac{1}{8}\frac{(-2)^3}{3} = -\frac{1}{3}$$

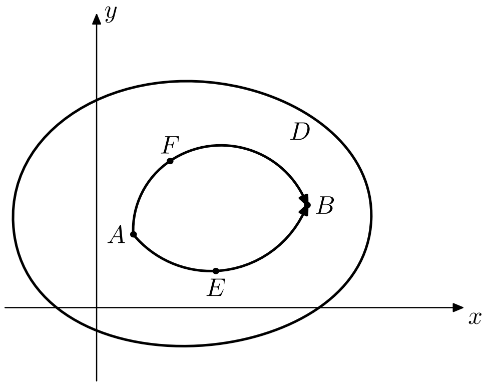
6.6 Path independent line integral

In this subsection we find out in what conditions the line integral

$$\int_{AB} X(x, y)dx + Y(x, y)dy \tag{6.16}$$

depends only on the endpoints A and B of the line but not on the path of integration.

Assume that in the region D containing the points A and B the functions $X(x, y)$ and $Y(x, y)$ and the partial derivatives $\dfrac{\partial X}{\partial y}$ and $\dfrac{\partial Y}{\partial x}$ are continuous. Let's choose two whatever curves AEB and AFB in the region D joining the points A and B.



So, we want to know in which conditions for any curves AEB and AFB

$$\int\limits_{AEB} Xdx + Ydy = \int\limits_{AFB} Xdx + Ydy$$

i.e.

$$\int\limits_{AEB} Xdx + Ydy - \int\limits_{AFB} Xdx + Ydy = 0$$

By Property 2 of the line integral with respect to coordinates

$$\int\limits_{AEB} Xdx + Ydy + \int\limits_{BFA} Xdx + Ydy = 0$$

and by Property 1

$$\int\limits_{AEBFA} Xdx + Ydy = 0$$

Denoting the closed curve $AEBFA = L$, we obtain the condition

$$\oint\limits_{L} Xdx + Ydy = 0 \tag{6.17}$$

This condition we obtain for any curves between any two points A and B in the region D. We shall call the curve joining the points A and B the *path of integration*.

Consequently, if the line integral (6.16) is path independent, then for each closed curve L in the region D there holds (6.17).

Theorem 1. The line integral (6.16) is path independent in the region D if and only if for any closed curve L in the region D there holds (6.17).

Next, suppose that for every closed curve L in the region D there holds (6.17). By the assumptions made in the beginning of this subsection there holds Green's formula. Denote by Δ the region enclosed by the closed curve L. According to Green's formula (6.15)

$$\iint_{\Delta} \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) dxdy = 0$$

Then also

$$\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} = 0$$

or

$$\frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y} \tag{6.18}$$

Now Theorem 1 gives us the following theorem.

Theorem 2. The line integral (6.16) is path independent in the region D if and only if in the region D there holds the condition (6.18).

The path independent line integral (6.16) is also denoted by

$$\int_A^B X dx + Y dy$$

Example 1. The line integral

$$\int_A^B (2x \cos y - y^2 \sin x) dx + (2y \cos x - x^2 \sin y) dy$$

is path independent because

$$\frac{\partial}{\partial x}(2y \cos x - x^2 \sin y) = -2y \sin x - 2x \sin y$$

and

$$\frac{\partial}{\partial y}(2x \cos y - y^2 \sin x) = -2x \sin y - 2y \sin x$$

Example 2. Compute

$$\int_{(0,0)}^{(2,1)} 2xy dx + x^2 dy$$

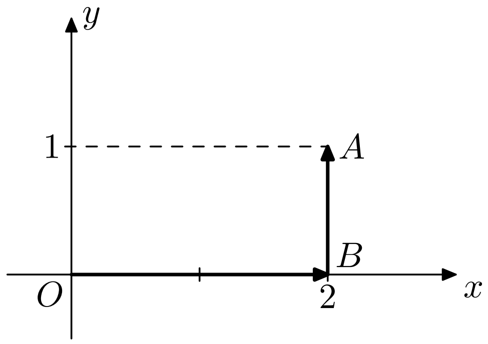
This line integral is path independent because

$$\frac{\partial(x^2)}{\partial x} = 2x$$

and

$$\frac{\partial(2xy)}{\partial y} = 2x$$

Thus, we can choose whatever path of integration joining the points $(0;0)$ and $(2;1)$. Let's choose the broken line OBA, where $O(0,0)$, $B(2;0)$ and $A(2;1)$. Usually, choosing the kind of broken line, whose segments are parallel to coordinate axes, gives us the most simple computation.



By Property 1 of the line integral with respect to coordinates

$$\int\limits_{(0,0)}^{(2,1)} 2xydx + x^2dy = \int\limits_{(0,0)}^{(2,0)} 2xydx + x^2dy + \int\limits_{(2,0)}^{(2,1)} 2xydx + x^2dy$$

The equation of the line OB is $y = 0$, which gives $y' = 0$. On the segment OB $0 \leq x \leq 2$ and by the formula (6.13)

$$\int\limits_{(0,0)}^{(2,0)} 2xydx + x^2dy = \int\limits_{0}^{2} (2x \cdot 0 + x^2 \cdot 0)dx = 0$$

The equation of the line BA is $x = 2$, i.e. $x' = 0$. On the segment BA the variable $0 \leq y \leq 1$ and by the formula (6.14)

$$\int\limits_{(2,0)}^{(2,1)} 2xydx + x^2dy = \int\limits_{0}^{1} (4y \cdot 0 + 4)dy = 4$$

Hence,

$$\int\limits_{(0,0)}^{(2,1)} 2xydx + x^2dy = 4$$

If there exists a function of two variables $u(x, y)$ such that the total differential of this function is

$$du = X(x, y)dx + Y(x, y)dy$$

i.e. $X = \dfrac{\partial u}{\partial x}$ and $Y = \dfrac{\partial u}{\partial y}$, then

$$\frac{\partial X}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}$$

and

$$\frac{\partial Y}{\partial x} = \frac{\partial^2 u}{\partial y \partial x}$$

Because of continuity the condition (6.18) holds.

Recall that the vector field $\overrightarrow{F} = (X(x, y), Y(x, y))$ is conservative, if $\overrightarrow{F}$ is the gradient of a scalar field $u(x, y)$ and the function $u(x, y)$ is the potential function of $\overrightarrow{F}$. Then $du = X(x, y)dx + Y(x, y)dy$ is the total differential of $u(x, y)$ and the condition (6.18) holds.

Conclusion 1. For the conservative vector field $\overrightarrow{F} = (X(x, y), Y(x, y))$ the line integral (6.16) is path independent.

Conclusion 2. For the conservative vector field $\overrightarrow{F} = (X(x, y), Y(x, y))$ the line integral over any closed curve L

$$\oint\limits_L X(x, y)dx + Y(x, y)dy = 0$$

Conclusion 3. If $u(x, y)$ is the potential function of the conservative vector field $\overrightarrow{F} = (X(x, y), Y(x, y))$, then

$$\int\limits_A^B X(x, y)dx + Y(x, y)dy = \int\limits_A^B du(x, y) = u(x, y)\Big|_A^B$$