Evaluate the double integral

1. $\int\limits_0^a dy \int\limits_{y-a}^{2y} xydx$

$$\int\limits_{y-a}^{2y} xydx = y\int\limits_{y-a}^{2y} xdx = y \cdot \frac{x^2}{2}\Big|_{y-a}^{2y} = y\left(\frac{4y^2}{2} - \frac{(y-a)^2}{2}\right) =$$
$$= y\left(\frac{4y^2}{2} - \frac{y^2}{2} + ay - \frac{a^2}{2}\right) = \frac{3}{2}y^3 + ay^2 - \frac{a^2}{2}y$$

$$\int\limits_0^a \left(\frac{3}{2}y^3 + ay^2 - \frac{a^2}{2}y\right) dy = \left(\frac{3}{2} \cdot \frac{y^4}{4} + a \cdot \frac{y^3}{3} - \frac{a^2}{2} \cdot \frac{y^2}{2}\right)\Big|_0^a =$$
$$= \frac{3a^4}{8} + \frac{a^4}{3} - \frac{a^4}{4} = \frac{11a^4}{24}$$

2. $\int\limits_3^4 dx \int\limits_1^2 \frac{dy}{(x+y)^2}$

3. $\int\limits_0^{\pi} dx \int\limits_0^{\frac{\pi}{2}} x\sin(x+y)dy$

$$d(x+y) = dy$$

$$\int\limits_0^{\frac{\pi}{2}} x\sin(x+y)dy = x\int\limits_0^{\frac{\pi}{2}} \sin(x+y)d(x+y) = -x\cos(x+y)\Big|_0^{\frac{\pi}{2}} =$$
$$= -x(\cos(x+\frac{\pi}{2}) - \cos x) = -x(-\sin x - \cos x) = x(\sin x + \cos x)$$

$$\int_0^{\pi} x(\sin x + \cos x)dx = \dots \qquad \int_a^b udv = uv\Big|_a^b - \int_a^b vdu$$

$$\begin{array}{ll} u = x & dv = (\sin x + \cos x)dx \\ du = dx & v = -\cos x + \sin x \end{array}$$

$$\dots = x(\sin x - \cos x)\Big|_0^{\pi} - \int_0^{\pi} (\sin x - \cos x)dx =$$

$$= \pi + (\cos x + \sin x)\Big|_0^{\pi} = \pi - 1 - 1 = \pi - 2$$

Sketch the region of integration

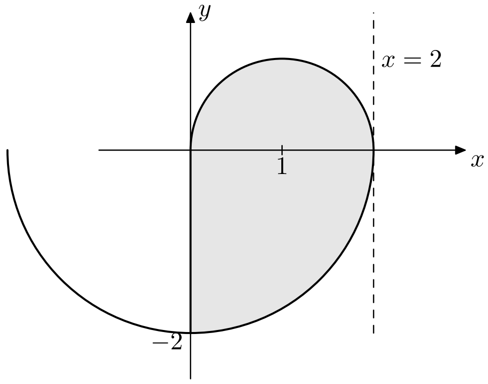
4. $\int_{-1}^{2} dy \int_{-y^2}^{y^2} f(x,y)dx$

5. $\int_{0}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{2x-x^2}} f(x,y)dy$

$$0 \leq x \leq 2 \qquad -\sqrt{4-x^2} \leq y \leq \sqrt{2x-x^2}$$

$$y = -\sqrt{4-x^2} \Rightarrow y^2 = 4 - x^2 \Rightarrow x^2 + y^2 = 4$$

$$y = \sqrt{2x-x^2} \Rightarrow y^2 = 2x - x^2 \Rightarrow x^2 - 2x + 1 + y^2 = 1 \Rightarrow (x-1)^2 + y^2 = 1$$



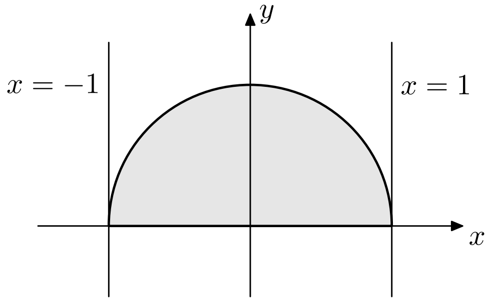
Sketch the region of integration and reverse the order of integration

6. $\int\limits_{-1}^{1} dx \int\limits_{0}^{\sqrt{1-x^2}} f(x,y)dy$

$$-1 \le x \le 1 \quad 0 \le y \le \sqrt{1-x^2}$$

$$x=-1, \quad x=1, \quad y=0, \quad y=\sqrt{1-x^2}$$

$$y=\sqrt{1-x^2} \quad \Rightarrow \quad y^2 = 1-x^2 \quad \Rightarrow \quad x^2+y^2=1$$



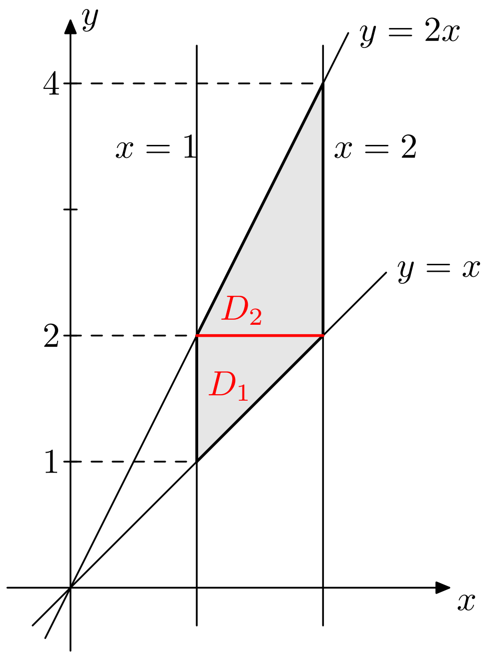
$$x^2+y^2=1 \quad \Rightarrow \quad x^2=1-y^2 \quad \Rightarrow \quad x=\pm\sqrt{1-y^2}$$

$$D: \quad \begin{array}{l} 0 \le y \le 1 \\ -\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \end{array}$$

$$\int\limits_{-1}^{1} dx \int\limits_{0}^{\sqrt{1-x^2}} f(x,y)dy = \int\limits_{0}^{1} dy \int\limits_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)dx$$

7. $\int\limits_{1}^{2} dx \int\limits_{x}^{2x} f(x,y)dy$

$$D: \quad 1 \le x \le 2 \quad x \le y \le 2x$$
$$x = 1, \quad x = 2, \quad y = x, \quad y = 2x$$



$$D_1: 1 \le y \le 2 \qquad\qquad D_2: 2 \le y \le 4$$
$$1 \le x \le y \qquad\qquad \frac{y}{2} \le x \le 2$$

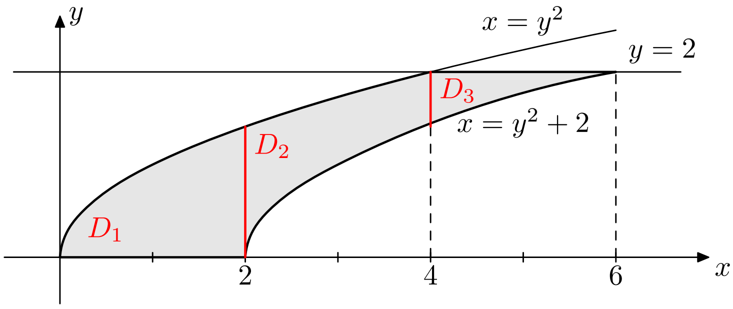
$$\int\limits_{1}^{2} dx \int\limits_{x}^{2x} f(x,y)dy = \int\limits_{1}^{2} dy \int\limits_{1}^{y} f(x,y)dx + \int\limits_{2}^{4} dy \int\limits_{\frac{y}{2}}^{2} f(x,y)dx$$

8. $\int\limits_{-2}^{2} dy \int\limits_{-1}^{y^2+1} f(x,y)dx$

9. $\int\limits_0^2 dy \int\limits_{y^2}^{y^2+2} f(x,y)dx$

$0 \le y \le 2 \qquad\qquad y^2 \le x \le y^2+2$

$y=0, \quad y=2, \quad x=y^2, \quad x=y^2+2$



$$x=y^2 \quad \Rightarrow \quad y=\pm\sqrt{x} \quad \Rightarrow \quad y=\sqrt{x}$$
$$x=y^2+2 \quad \Rightarrow \quad y=\pm\sqrt{x-2} \quad \Rightarrow \quad y=\sqrt{x-2}$$

$D_1: 0 \le x \le 2$, $\quad 0 \le y \le \sqrt{x}$

$D_2: 2 \le x \le 4$, $\quad \sqrt{x-2} \le y \le \sqrt{x}$

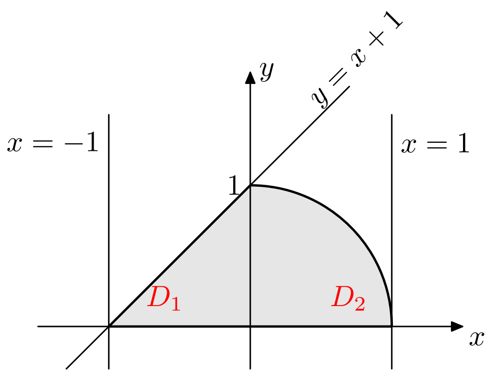
$D_3: 4 \le x \le 6$, $\quad \sqrt{x-2} \le y \le 2$

$$\int\limits_0^2 dy \int\limits_{y^2}^{y^2+2} f(x,y)dx = \int\limits_0^2 dx \int\limits_0^{\sqrt{x}} f(x,y)dy +$$
$$+\int\limits_2^4 dx \int\limits_{\sqrt{x-2}}^{\sqrt{x}} f(x,y)dy + \int\limits_4^6 dx \int\limits_{\sqrt{x-2}}^{2} f(x,y)dy$$

10. $\int\limits_{-1}^{0} dx \int\limits_{0}^{x+1} f(x,y)dy + \int\limits_{0}^{1} dx \int\limits_{0}^{\sqrt{1-x^2}} f(x,y)dy$

$$D_1: \begin{array}{l} -1 \le x \le 0 \\ 0 \le y \le x+1 \end{array} \qquad D_2: \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le \sqrt{1-x^2} \end{array}$$

$$x = -1, \quad x = 0, \quad x = 1, \quad y = 0, \quad y = x+1, \quad y = \sqrt{1-x^2}$$



$$y = x + 1 \quad \Rightarrow \quad x = y - 1$$

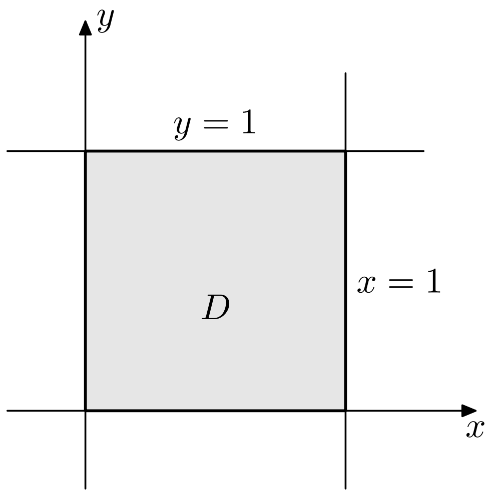
$$y = \sqrt{1-x^2} \ \Rightarrow \ x^2 + y^2 = 1 \ \Rightarrow \ x^2 = 1 - y^2 \ \Rightarrow \ x = \sqrt{1-y^2}$$

$$D: \quad 0 \le y \le 1 \quad y - 1 \le x \le \sqrt{1-y^2}$$

$$\int\limits_{-1}^{0} dx \int\limits_{0}^{x+1} f(x,y)dy + \int\limits_{0}^{1} dx \int\limits_{0}^{\sqrt{1-x^2}} f(x,y)dy = \int\limits_{0}^{1} dy \int\limits_{y-1}^{\sqrt{1-y^2}} f(x,y)dx$$

Sketch the region of integration and evaluate the double integral

11. $\iint\limits_D \frac{x^2}{1+y^2}dxdy$ if D the quadrat $0 \le x \le 1$ and $0 \le y \le 1$.

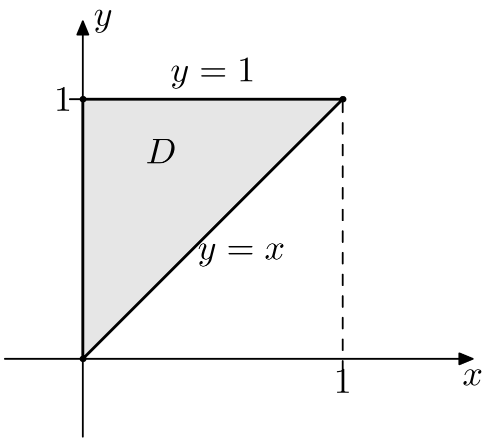


$$\iint\limits_D \frac{x^2}{1+y^2}dxdy = \int\limits_0^1 dx \int\limits_0^1 \frac{x^2}{1+y^2}dy$$

$$\int\limits_0^1 \frac{x^2}{1+y^2}dy = x^2 \int\limits_0^1 \frac{dy}{1+y^2} = x^2 \cdot \arctan y\Big|_0^1 = x^2 \cdot \frac{\pi}{4}$$

$$\int\limits_0^1 dx \int\limits_0^1 \frac{x^2}{1+y^2}dy = \int\limits_0^1 \frac{\pi}{4}x^2 dx = \frac{\pi}{4} \cdot \frac{x^3}{3}\Big|_0^1 = \frac{\pi}{12}$$

12. $\iint\limits_D x dx dy$ if D is the triangle with vertices $(0;0)$, $(1;1)$ and $(0;1)$.



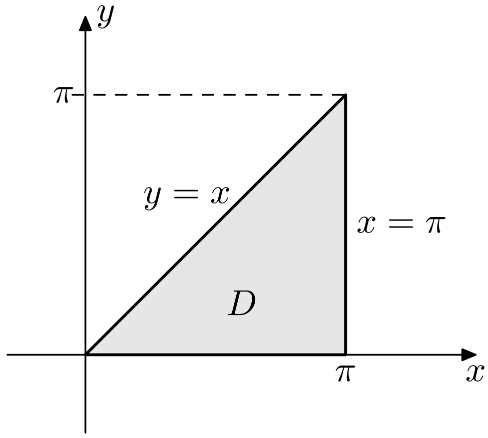
$$D: \begin{array}{l} 0 \le x \le 1 \\ x \le y \le 1 \end{array} \qquad D: \begin{array}{l} 0 \le y \le 1 \\ 0 \le x \le y \end{array}$$

$$\iint\limits_D x dx dy = \int\limits_0^1 dy \int\limits_0^y x dx$$

$$\int\limits_0^y x dx = \left.\frac{x^2}{2}\right|_0^y = \frac{y^2}{2}$$

$$\int\limits_0^1 dy \int\limits_0^y x dx = \frac{1}{2}\int\limits_0^1 y^2 dy = \frac{1}{2} \cdot \left.\frac{y^3}{3}\right|_0^1 = \frac{1}{6}$$

13. $\iint\limits_D \cos(x-y)dxdy$ if D the triangle bounded by $y = 0$, $y = x$ and $x = \pi$.



$$D: \quad \begin{array}{l} 0 \le x \le \pi \\ 0 \le y \le x \end{array}$$

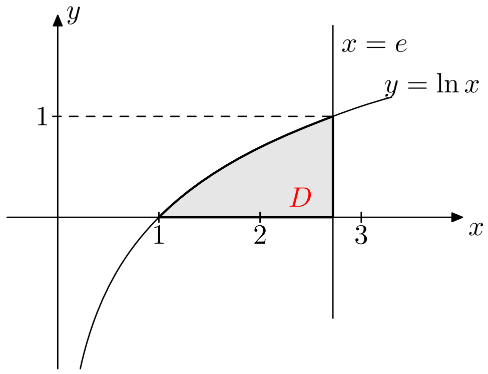
$$\iint\limits_D \cos(x-y)dxdy = \int\limits_0^\pi dx \int\limits_0^x \cos(x-y)dy$$

$$d(x-y) = -dy \quad \Rightarrow \quad dy = -d(x-y)$$

$$\int\limits_0^x \cos(x-y)dy = -\int\limits_0^x \cos(x-y)d(x-y) = -\sin(x-y)\Big|_0^x = \sin x$$

$$\int\limits_0^\pi \sin x dx = -\cos x\Big|_0^\pi = -\cos\pi + \cos 0 = 2$$

14. $\displaystyle\iint\limits_D \frac{1}{x}dxdy$ if D is bounded by $y = 0$, $y = \ln x$ and $x = e$.



$$D: \quad \begin{array}{l} 1 \le x \le e \\ 0 \le y \le \ln x \end{array}$$

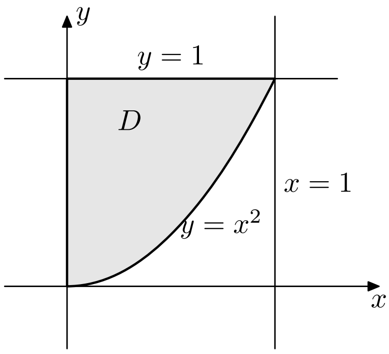
$$\iint\limits_D \frac{1}{x}dxdy = \int\limits_1^e dx \int\limits_0^{\ln x} \frac{1}{x}dy$$

$$\int\limits_0^{\ln x} \frac{1}{x}dy = \frac{1}{x}\int\limits_0^{\ln x} dy = \frac{1}{x}\cdot y\Big|_0^{\ln x} = \frac{1}{x}\cdot \ln x = \frac{\ln x}{x}$$

$$\int\limits_1^e \frac{\ln x}{x}dx = \int\limits_1^e \ln x\frac{dx}{x}\int\limits_1^e \ln x d(\ln x) = \frac{\ln^2 x}{2}\Big|_1^e = \frac{1}{2}$$

15. $\int\limits_0^1 dx \int\limits_{x^2}^1 xe^{y^2} dy$

$$D: \quad \begin{array}{l} 0 \le x \le 1 \\ x^2 \le y \le 1 \end{array}$$



$$D: \quad \begin{array}{l} 0 \le y \le 1 \\ 0 \le x \le \sqrt{y} \end{array}$$

$$\int\limits_0^1 dx \int\limits_{x^2}^1 xe^{y^2} dy = \int\limits_0^1 dy \int\limits_0^{\sqrt{y}} xe^{y^2} dx$$

$$\int\limits_0^{\sqrt{y}} xe^{y^2} dx = e^{y^2} \int\limits_0^{\sqrt{y}} xdx = e^{y^2} \cdot \left.\frac{x^2}{2}\right|_0^{\sqrt{y}} = e^{y^2} \cdot \frac{y}{2}$$

$$\frac{1}{2} \int\limits_0^1 e^{y^2} y dy$$

$$d(y^2) = 2ydy \quad \Rightarrow \quad ydy = \frac{1}{2} d(y^2)$$

$$\frac{1}{2} \cdot \frac{1}{2} \int\limits_0^1 e^{y^2} d(y^2) = \frac{1}{4} e^{y^2} \Big|_0^1 = \frac{1}{4}(e-1) = \frac{e-1}{4}$$