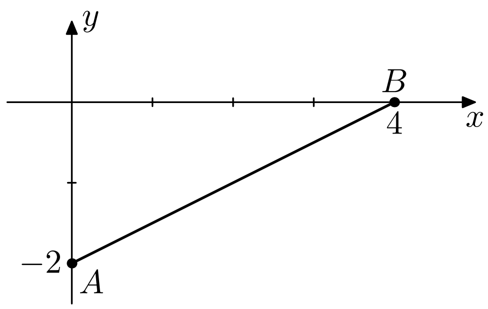
Line integral with respect to length of arc

$$AB: \quad y = y(x), \quad A(a; y(a)), \quad B(b; y(b))$$

$$\int\limits_{AB} f(x,y)ds = \int\limits_a^b f(x, y(x))\sqrt{1+y'^2}dx$$

1. Evaluate $\int\limits_L \frac{ds}{x-y}$ if L is the segment of the line $y = \frac{x}{2} - 2$ from the point $A(0; -2)$ to $B(4; 0)$



$$y' = \frac{1}{2} \quad \Rightarrow \quad 1 + y'^2 = 1 + \frac{1}{4} = \frac{5}{4} \quad \Rightarrow \quad \sqrt{1+y'^2} = \frac{\sqrt{5}}{2}$$

$$\int\limits_L \frac{ds}{x-y} = \int\limits_0^4 \frac{\frac{\sqrt{5}}{2}dx}{x - (\frac{x}{2} - 2)} = \sqrt{5}\int\limits_0^4 \frac{dx}{2(\frac{x}{2}+2)} =$$

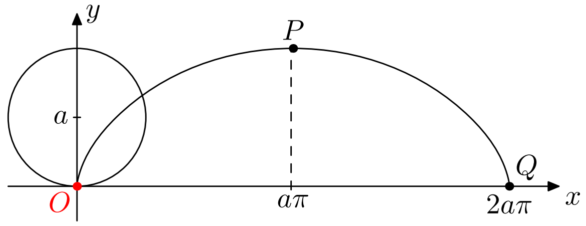
$$= \sqrt{5}\int\limits_0^4 \frac{d(x+4)}{x+4} = \sqrt{5}\ln|x+4|\Big|_0^4 = \sqrt{5}(\ln 8 - \ln 4) = \sqrt{5}\ln 2$$

$$l_{AB} = \int\limits_{AB} ds$$

2. Evaluate the length of the first arc of cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ $(0 \leq t \leq 2\pi)$.

$$AB: \quad x = x(t),\ y = y(t), \quad A(x(\alpha); y(\alpha)), \quad B(x(\beta); y(\beta))$$

$$\int\limits_{AB} f(x,y)ds = \int\limits_{\alpha}^{\beta} f(x(t), y(t))\sqrt{\dot{x}^2 + \dot{y}^2}dt$$



$$\dot{x} = a(1 - \cos t), \quad \dot{y} = a \sin t$$

$$\dot{x}^2 + \dot{y}^2 = a^2(1 - \cos t)^2 + a^2 \sin^2 t = a^2(1 - 2\cos t + \cos^2 t + \sin^2 t) = 2a^2(1 - \cos t)$$

$$l = \int\limits_0^{2\pi} \sqrt{2a^2(1 - \cos t)}dt$$

$$1 - \cos t = 2\sin^2 \frac{t}{2}$$

$$0 \leq t \leq 2\pi \quad \Rightarrow \quad 0 \leq \frac{t}{2} \leq \pi \quad \Rightarrow \quad \sin \frac{t}{2} \geq 0$$

$$l = \int\limits_0^{2\pi} \sqrt{2a^2 \cdot 2\sin^2 \frac{t}{2}}dt = 2a \int\limits_0^{2\pi} \sin \frac{t}{2} dt = 4a \int\limits_0^{2\pi} \sin \frac{t}{2} d\left(\frac{t}{2}\right) =$$

$$= -4a \cos \frac{t}{2}\Big|_0^{2\pi} = -4a(-1 - 1) = 8a$$

$$AB:\ \ x=x(t),\ y=y(t),\ z=z(t)\quad A:\ t=\alpha,\quad B:\ t=\beta$$

$$\int\limits_{AB} f(x,y,z)ds=\int\limits_{\alpha}^{\beta} f(x(t),y(t),z(t))\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}dt$$

3. Evaluate $\int\limits_{L}\dfrac{z^2ds}{x^2+y^2}$ if L the first arc of the screw line $x=a\cos t,\ y=a\sin t,\ z=t.$

$$0\le t\le 2\pi$$

$$\dot{x}=-a\sin t,\quad \dot{y}=a\cos t,\ \ \dot{z}=1$$

$$\dot{x}^2+\dot{y}^2+\dot{z}^2=a^2\sin^2 t+a^2\cos^2 t+1=a^2+1$$

$$\int\limits_{L}\frac{z^2ds}{x^2+y^2}=\int\limits_{0}^{2\pi}\frac{t^2\sqrt{a^2+1}dt}{a^2\cos^2 t+a^2\sin^2 t}=$$

$$=\frac{\sqrt{a^2+1}}{a^2}\int\limits_{0}^{2\pi}t^2dt=\frac{\sqrt{a^2+1}}{a^2}\cdot\frac{t^3}{3}\Bigg|\int\limits_{0}^{2\pi}=\frac{8\pi^3\sqrt{a^2+1}}{3a^2}$$

Line integral with respect to coordinates

$$AB: \quad y = y(x), \quad A(a; y(a)), \quad B(b; y(b))$$

$$\int\limits_{AB} X(x,y)dx + Y(x,y)dy = \int\limits_a^b [X(x,y(x)) + Y(x,y(x))y']dx$$

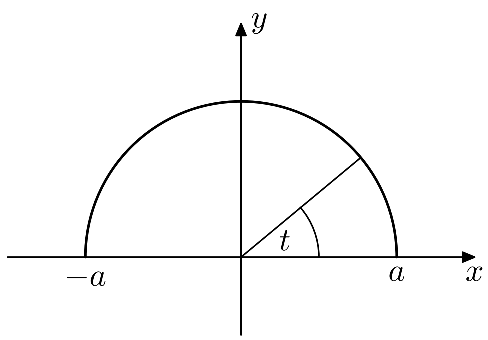
4. Evaluate $\int\limits_L (x^2 - y^2)dx$ if L is the arc of parabola $y = x^2$ from $(0;0)$ to $(2;4)$.

$$\int\limits_L (x^2 - y^2)dx = \int\limits_0^2 (x^2 - x^4)dx = \frac{x^3}{3}\bigg|_0^2 - \frac{x^5}{5}\bigg|_0^2 = \frac{8}{3} - \frac{32}{5} = -\frac{56}{15}$$

$$AB: \quad x = x(t), \ y = y(t), \quad A(x(\alpha); y(\alpha)), \quad B(x(\beta); y(\beta))$$

$$\int\limits_{AB} X(x,y)dx + Y(x,y)dy = \int\limits_\alpha^\beta [X(x(t),y(t))\dot{x} + Y(x(t),y(t))\dot{y}]dt$$

5. Evaluate $\int\limits_L \frac{y^2dx - x^2dy}{x^2 + y^2}$, if L is the half circle $x = a\cos t$, $y = a\sin t$ $(0 \le t \le \pi)$.



$$\dot{x} = -a\sin t, \ \dot{y} = a\cos t$$

$$\int\limits_L \frac{y^2dx - x^2dy}{x^2+y^2} = \int\limits_0^\pi \frac{a^2\sin^2 t(-a\sin t) - a^2\cos^2 t \cdot a\cos t}{a^2\cos^2 t + a^2\sin^2 t}dt =$$

$$= \int\limits_0^\pi \frac{-a^3(\sin^3 t + \cos^3 t)}{a^2}dt = -a\int\limits_0^\pi (\sin^3 t + \cos^3 t)dt$$

$$\int\limits_0^\pi \sin^3 tdt = \int\limits_0^\pi \sin^2 t \sin tdt = \int\limits_0^\pi (1-\cos^2 t)\sin tdt = ...$$

$$u = \cos t \quad du = -\sin tdt \quad \sin tdt = -du$$

$$... = \int\limits_1^{-1}(1-u^2)(-du) = \int\limits_{-1}^{1}(1-u^2)du = \left(u - \frac{u^3}{3}\right)\Bigg|_{-1}^{1} = 1-\frac{1}{3}-(-1+\frac{1}{3}) = \frac{4}{3}$$

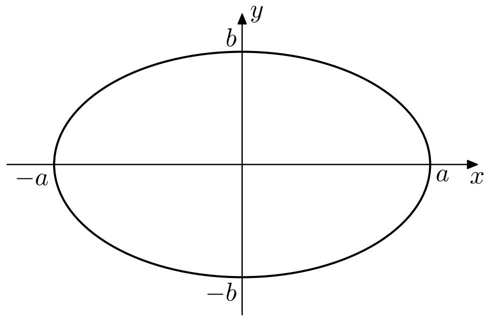
$$\int\limits_0^\pi \cos^3 tdt = \int\limits_0^\pi (1-\sin^2 t)\cos tdt = ...$$

$$u = \sin t \quad du = \cos tdt$$

$$... = \int\limits_0^0 (1-u^2)du = 0$$

$$-a\int\limits_0^\pi (\sin^3 t + \cos^3 t)dt = -\frac{4a}{3}$$

6. Evaluate $\int\limits_L ydx - xdy$ if L is the ellipse $x = a\cos t,\ y = b\sin t$ traversed in positive direction.



$$\dot{x} = -a\sin t,\ \dot{y} = b\cos t$$

$$\int\limits_L ydx - xdy = \int\limits_0^{2\pi} (b\sin t \cdot (-a\sin t) - at \cdot b\cos t)dt =$$

$$= -ab\int\limits_0^{2\pi} (\sin^2 t + \cos^2 t)dt = -ab\int\limits_0^{2\pi} dt = -ab \cdot t\Big|_0^{2\pi} = -2\pi ab$$

$$AB:\ \ x = x(t),\ y = y(t),\ z = z(t) \quad A:\ t = \alpha, \quad B:\ t = \beta$$

$$\int\limits_{AB} X(x,y,z)dx + Y(x,y,z)dy + Z(x,y,z)dz =$$

$$= \int\limits_\alpha^\beta [X(x(t),y(t),z(t))\dot{x} + Y(x(t),y(t),z(t))\dot{y} + Z(x(t),y(t),z(t))\dot{z}]dt$$

7. Evaluate $\int\limits_L xdx + ydy + (x+y-1)dz$ if L is the line segment from $(1;1;1)$ to $(2;3;4)$.

$$\vec{s} = (1;2;3)$$

$$x = 1 + t, \quad y = 1 + 2t, \quad z = 1 + 3t$$

$$(1; 1, 1) \;\; t = 0, \quad (2; 3; 4) \;\; t = 1$$

$$\dot{x} = 1, \quad \dot{y} = 2, \quad \dot{z} = 3$$

$$\int\limits_L xdx + ydy + (x + y - 1)dz =$$

$$= \int\limits_0^1 [(1 + t) \cdot 1 + (1 + 2t) \cdot 2 + (1 + t + 1 + 2t - 1) \cdot 3]dt =$$

$$= \int\limits_0^1 (14t + 6)dt = (7t^2 + 6t)\Big|_0^1 = 13$$

8. Evaluate $\int\limits_L yzdx + zxdy + xydz$ if L is the arc of screw line $x = R\cos t$, $y = R\sin t$, $z = \dfrac{at}{2\pi}$ between the planes $z = 0$ and $z = a$.

$$z = 0 \;\; \Rightarrow \;\; \frac{at}{2\pi} = 0 \;\; \Rightarrow \;\; t = 0$$

$$z = a \;\; \Rightarrow \;\; \frac{at}{2\pi} = a \;\; \Rightarrow \;\; \frac{t}{2\pi} = 1 \;\; \Rightarrow \;\; t = 2\pi$$

$$\dot{x} = -R\sin t, \quad \dot{y} = R\cos t, \quad \dot{z} = \frac{a}{2\pi}$$

$$\int\limits_L yzdx + zxdy + xydz =$$

$$= \int\limits_0^{2\pi} \left[R\sin t \cdot \frac{at}{2\pi}(-R\sin t) + \frac{at}{2\pi} R\cos t \cdot R\cos t + R\cos t \cdot R\sin t \cdot \frac{a}{2\pi} \right] dt$$

$$= \frac{aR^2}{2\pi} \int\limits_0^{2\pi} (-t\sin^2 t + t\cos^2 t + \sin t\cos t)dt =$$

$$= \frac{aR^2}{2\pi} \int\limits_0^{2\pi} (t\cos 2t + \sin t\cos t)dt$$

$$\int\limits_0^{2\pi} t\cos 2tdt = ...$$

$$u = t \qquad dv = \cos 2tdt$$

$$du = dt \qquad dv = \frac{1}{2}\sin 2tdt$$

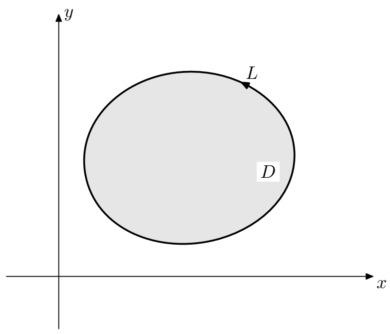
$$... = \frac{t}{2}\sin 2t\Big|_0^{2\pi} - \frac{1}{2}\int\limits_0^{2\pi} \sin 2tdt = -\frac{1}{4}\int\limits_0^{2\pi} \sin 2td(2t) = \frac{1}{4}\cos 2t\Big|_0^{2\pi} = \frac{1}{4}(1-1) = 0$$

$$\int\limits_0^{2\pi} \sin t\cos tdt = ...$$

$$u = \sin t \qquad du = \cos tdt$$

$$... = \int\limits_0^0 udu = 0$$

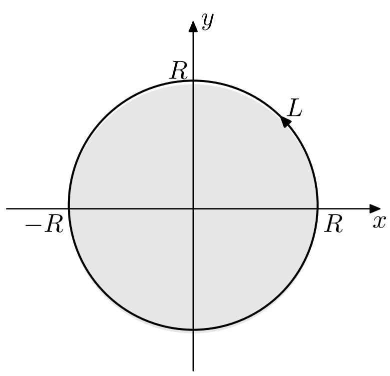
$$\frac{aR^2}{2\pi} \int\limits_0^{2\pi} (t\cos 2t + \sin t\cos t)dt = 0$$



Green's formula

$$\oint\limits_L X(x,y)dx + Y(x,y)dy = \iint\limits_D \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) dxdy$$

9. Evaluate $\oint\limits_L ydx - xdy$ if L is the positively oriented circle $x^2 + y^2 = R^2$.

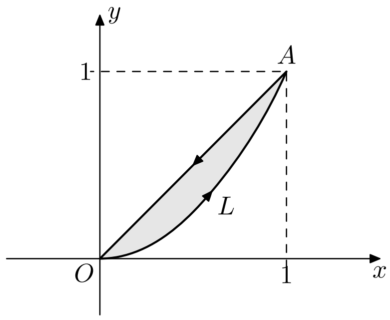


$$Y = -x \qquad X = y$$

$$\frac{\partial Y}{\partial x} = -1 \qquad \frac{\partial X}{\partial y} = 1 \qquad \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} = -2$$

$$\oint\limits_L ydx - xdy = \iint\limits_D (-2)dxdy = -2\iint\limits_D dxdy = -2\pi R^2$$

10. Evaluate $\oint\limits_L (x+y)^2dx-(x-y)^2dy$ if L is the closed curve, which goes along the arc of parabola $y=x^2$ from the point $O(0;0)$ to the point $A(1;1)$ and from A along the line $y=x$ back to $O(0;0)$.



$$X=(x+y)^2 \qquad Y=-(x-y)^2$$

$$\frac{\partial Y}{\partial x}=-2(x-y) \qquad \frac{\partial X}{\partial y}=2(x+y)$$

$$\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}=-2x+2y-2x-2y=-4x$$

$$\oint\limits_L (x+y)^2dx-(x-y)^2dy=-4\iint\limits_D xdxdy$$

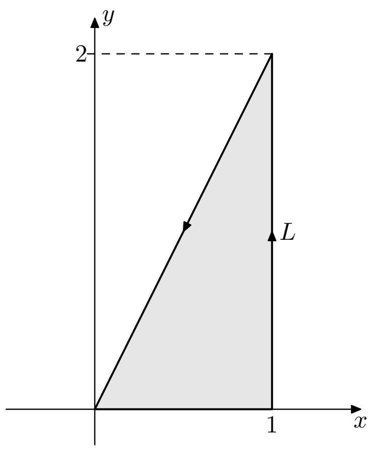
$$D: \quad 0\le x\le 1 \quad x^2\le y\le x$$

$$-4\iint\limits_D xdxdy=-4\int\limits_0^1 dx\int\limits_{x^2}^{x} xdy$$

$$\int\limits_{x^2}^{x} xdy=x\cdot y\Big|_{x^2}^{x}=x(x-x^2)=x^2-x^3$$

$$-4\int\limits_0^1(x^2-x^3)dx=-4\left(\frac{x^3}{3}-\frac{x^4}{4}\right)\Big|_0^1=-4\left(\frac{1}{3}-\frac{1}{4}\right)=-\frac{1}{3}$$

11. Evaluate $\oint\limits_L xydx + x^2y^3dy$ if L is the positively oriented contour of triangle with vertices $(0;0)$, $(1;0)$ and $(1;2)$.



$$X = xy \qquad Y = x^2y^3$$

$$\frac{\partial Y}{\partial x} = 2xy^3 \qquad \frac{\partial X}{\partial y} = x \qquad \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} = 2xy^3 - x$$

$$\oint\limits_L xydx + x^2y^3dy = \iint\limits_D (2xy^3 - x)dxdy$$

$$D: \quad 0 \le x \le 1 \quad 0 \le y \le 2x$$

$$\iint\limits_D (2xy^3 - x)dxdy = \int\limits_0^1 dx \int\limits_0^{2x} x(2y^3 - 1)dy$$

$$x\int\limits_0^{2x}(2y^3-1)dy = x\left(\frac{2y^4}{4} - y\right)\Bigg|_0^{2x} = x(8x^4-2x) = 8x^5-2x^2$$

$$\int\limits_0^1 (8x^5 - 2x^2)dx = \left(\frac{4x^6}{3} - \frac{2x^3}{3}\right)\Bigg|_0^1 = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$$

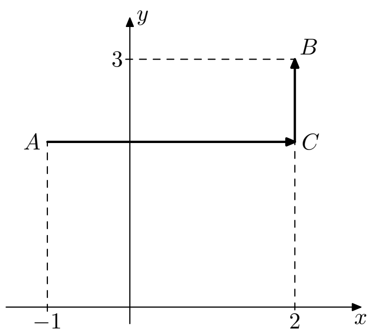
Path independent line integral

$$\int\limits_{AB} X(x,y)dx + Y(x,y)dy$$

$$\frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y}$$

12. Evaluate $\int\limits_{(-1;2)}^{(2;3)} ydx + xdy$

$$X = y \qquad Y = x \qquad \frac{\partial Y}{\partial x} = 1 \qquad \frac{\partial X}{\partial y} = 1$$



$$A(-1;2), \qquad C(2;2), \qquad B(2;3)$$

$$\int\limits_{A}^{B} ydx + xdy = \int\limits_{A}^{C} ydx + xdy + \int\limits_{C}^{B} ydx + xdy$$

$$AC: \quad y = 2, \qquad y' = 0, \qquad -1 \le x \le 2$$

$$\int\limits_{A}^{C} ydx + xdy = \int\limits_{-1}^{2} (2 + x \cdot 0)dx = 2x\Big|_{-1}^{2} = 2(2+1) = 6$$

$$CB: \quad x = 2, \qquad x' = 0, \qquad 2 \le y \le 3$$

$$\int_C^B ydx + xdy = \int_2^3 (y \cdot 0 + 2)dy = 2y\Big|_2^3 = 2(3-2) = 2$$

$$\int_{(-1;2)}^{(2;3)} ydx + xdy = 6 + 2 = 8$$

13. Evaluate $\displaystyle\int_{(0;0)}^{(1;1)} 2xydx + x^2dy$

$$X = 2xy \qquad Y = x^2 \qquad \frac{\partial Y}{\partial x} = 2x \qquad \frac{\partial X}{\partial y} = 2x$$

$$(0;0) \to \ (1;1) \quad y = x$$

$$\int_{(0;0)}^{(1;1)} 2xydx + x^2dy = \int_0^1 (2x \cdot x + x^2 \cdot 1)dx = \int_0^1 3x^2dx = x^3\Big|_0^1 = 1$$

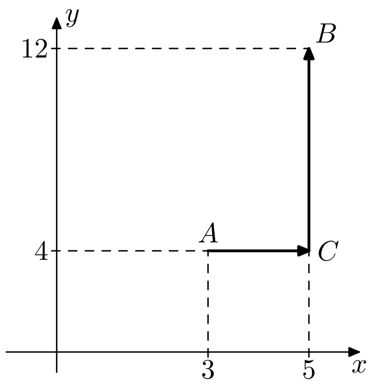
14. Evaluate $\displaystyle\int_{(3;4)}^{(5;12)} \frac{xdx + ydy}{x^2 + y^2}$

$$X = \frac{x}{x^2 + y^2} \qquad Y = \frac{y}{x^2 + y^2}$$

$$\frac{\partial Y}{\partial x} = -\frac{y}{(x^2 + y^2)^2} \cdot 2x = -\frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial X}{\partial y} = -\frac{x}{(x^2 + y^2)^2} \cdot 2y = -\frac{2xy}{(x^2 + y^2)^2}$$

$$A(3;4) \qquad B(5;12) \qquad C(5;4)$$



$$\int\limits_{(3;4)}^{(5;12)} \frac{xdx+ydy}{x^2+y^2} = \int\limits_A^C \frac{xdx+ydy}{x^2+y^2} + \int\limits_C^B \frac{xdx+ydy}{x^2+y^2}$$

$$AC: \quad y=4, \quad y'=0, \quad 3 \le x \le 5$$

$$\int\limits_A^C \frac{xdx+ydy}{x^2+y^2} = \int\limits_3^5 \frac{x+4\cdot 0}{x^2+16}dx = \int\limits_3^5 \frac{xdx}{x^2+16}$$

$$d(x^2+16)=2xdx \qquad xdx=\frac{1}{2}d(x^2+16)$$

$$\int\limits_3^5 \frac{xdx}{x^2+16} = \frac{1}{2}\int\limits_3^5 \frac{d(x^2+16)}{x^2+16} = \frac{1}{2}\ln(x^2+16)\Big|_3^5 = \frac{1}{2}(\ln 41 - \ln 25)$$

$$CB: \quad x=5, \quad x'=0, \quad 4 \le y \le 12$$

$$\int\limits_C^B \frac{xdx+ydy}{x^2+y^2} = \int\limits_4^{12} \frac{5\cdot 0+y}{25+y^2}dy = \int\limits_4^{12} \frac{ydy}{y^2+25}$$

$$d(y^2+25)=2ydy \qquad ydy=\frac{1}{2}d(y^2+25)$$

$$\int\limits_4^{12} \frac{ydy}{y^2+25} = \frac{1}{2}\int\limits_4^{12} \frac{d(y^2+25)}{y^2+25} = \frac{1}{2}\ln(y^2+25\Big|_4^{12} = \frac{1}{2}(\ln 169 - \ln 41)$$

$$\int_{(3;4)}^{(5;12)} \frac{xdx+ydy}{x^2+y^2} = \frac{1}{2}(\ln 41 - \ln 25) + \frac{1}{2}(\ln 169 - \ln 41) = \frac{1}{2}\ln\frac{169}{25} = \ln\frac{13}{5}$$

$$\frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y}$$

Vector field $(X(x,y); Y(x,y))$ is called conservative

$$\exists \ u(x,y): \quad du = X(x,y)dx + Y(x,y)dy$$

$u(x,y)$ is called potential function of vector field $(X(x,y); Y(x,y))$

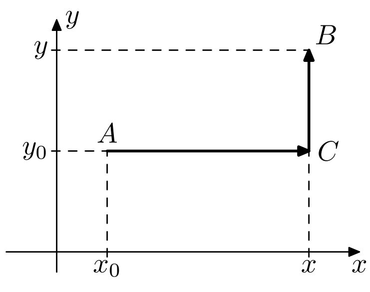
$$\int_{(x_0,y_0)}^{(x,y)} du = u\Big|_{(x_0,y_0)}^{(x,y)} = u(x,y) - u(x_0,y_0)$$

15. Find the function $u(x,y)$ if $du = x^2dx + y^2dy$

$$X = x^2 \qquad Y = y^2$$

$$\frac{\partial Y}{\partial x} = 0, \qquad \frac{\partial X}{\partial y} = 0$$

$$u(x,y) = \int_{(x_0,y_0)}^{(x,y)} x^2dx + y^2dy$$



$$A(x_0,y_0) \qquad B(x,y) \qquad C(x,y_0)$$

$$u(x,y)=\int_A^C x^2dx+y^2dy+\int_C^B x^2dx+y^2dy$$

$$AC: \quad y=y_0, \quad y'=0, \quad x_0\le x\le x$$

$$\int_A^C x^2dx+y^2dy=\int_{x_0}^{x} x^2dx=\left.\frac{x^3}{3}\right|_{x_0}^{x}=\frac{x^3}{3}-\frac{x_0^3}{3}$$

$$CB: \quad x=x, \quad x'=0, \quad y_0\le y\le y$$

$$\int_C^B x^2dx+y^2dy=\int_{y_0}^{y} y^2dy=\left.\frac{y^3}{3}\right|_{y_0}^{y}=\frac{y^3}{3}-\frac{y_0^3}{3}$$

$$u(x,y)=\frac{x^3}{3}-\frac{x_0^3}{3}+\frac{y^3}{3}-\frac{y_0^3}{3}$$

$$C=-\frac{x_0^3}{3}-\frac{y_0^3}{3}$$

$$u(x,y)=\frac{x^3}{3}+\frac{y^3}{3}+C$$

16. Find the function $u(x,y)$ if $du=(2x\cos y-y^2\sin x)dx+(2y\cos x-x^2\sin y)dy$

Let us use the sketch of previous exercise.

$$X=2x\cos y-y^2\sin x \qquad Y=2y\cos x-x^2\sin y$$

$$\frac{\partial Y}{\partial x}=-2y\sin x-2x\sin y \qquad \frac{\partial X}{\partial y}=-2x\sin y-2y\sin x$$

$$u(x,y)=\int_A^C(2x\cos y-y^2\sin x)dx+(2y\cos x-x^2\sin y)dy+$$

$$+\int_C^B(2x\cos y-y^2\sin x)dx+(2y\cos x-x^2\sin y)dy$$

$$AC: \quad y = y_0, \quad y' = 0, \quad x_0 \le x \le x$$

$$\int\limits_A^C (2x\cos y - y^2 \sin x)dx + (2y\cos x - x^2\sin y)dy =$$

$$= \int\limits_{x_0}^{x} (2x\cos y_0 - y_0^2 \sin x)dx = (x^2\cos y_0 + y_0^2\cos x)\Big|_{x_0}^{x} =$$

$$= x^2\cos y_0 + y_0^2\cos x - x_0^2\cos y_0 - y_0^2\cos x_0$$

$$CB: \quad x = x, \quad x' = 0, \quad y_0 \le y \le y$$

$$\int\limits_C^B (2x\cos y - y^2 \sin x)dx + (2y\cos x - x^2\sin y)dy =$$

$$= \int\limits_{y_0}^{y} (2y\cos x - x^2 \sin y)dy = (y^2\cos x + x^2\cos y)\Big|_{y_0}^{y} =$$

$$= y^2\cos x + x^2\cos y - y_0^2\cos x - x^2\cos y_0$$

$$\begin{aligned} u(x,y) = {} & x^2\cos y_0 + y_0^2\cos x - x_0^2\cos y_0 - y_0^2\cos x_0 + \\ & + y^2\cos x + x^2\cos y - y_0^2\cos x - x^2\cos y_0 = \\ & = y^2\cos x + x^2\cos y - x_0^2\cos y_0 - y_0^2\cos x_0 \end{aligned}$$

$$C = -x_0^2\cos y_0 - y_0^2\cos x_0$$

$$u(x,y) = y^2\cos x + x^2\cos y + C$$