Partial derivatives

Find partial derivatives with respect to each independent variable

1. $z = x^3y - y^3x$

2. $z = x\sqrt{y} + \frac{y}{\sqrt[3]{x}}$

3. $z = \frac{x^3 + y^3}{x^2 + y^2}$

4. $z = \ln(x^2 + y^2)$

5. $z = y^x$

6. $z = \arctan \frac{x}{y}$

7. $z = \ln(x + \sqrt{x^2 + y^2})$

8. $z = e^{-\frac{x}{y}}$

9. $z = \sin \frac{x}{y} \cos \frac{y}{x}$

10. $z = \arcsin(y^2\sqrt{x})$

11. $u = x^3 + yz^2 + 3yx - x + z$

12. $w = \sqrt{x^2 + y^2 + z^2}$

13. $w = x^{\frac{y}{z}}$

14. $w = x^{y^z}$

15. Let $z = \ln(x^2 + xy + y^2)$. Prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2$

16. Let $z = xy + xe^{\frac{y}{x}}$. Prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy + z$

17. Let $u = (x - y)(y - z)(z - x)$. Prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

18. Let $w = x + \frac{x - y}{y - z}$. Prove that $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 1$

Total differential

Total differential of function $z = f(x, y)$ is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

19. $z = \sin(xy)$

$$dz = \cos(xy)ydx + \cos(xy)xdy = \cos(xy)(ydx + xdy)$$

20. $z = \arcsin \dfrac{x}{y}$

Total differential of function $w = f(x, y, z)$ is

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

21. $w = x^{yz}$

$$dw = yz \cdot x^{yz-1}dx + x^{yz}\ln x \cdot z dy + x^{yz}\ln x \cdot y dz = x^{yz}\left(\frac{yz}{x}dx + z\ln x dy + y\ln x dz\right)$$

22. Evaluate dz if $z = x + y - \sqrt{x^2 + y^2}$, $x = 3$, $y = 4$, $\Delta x = 0,1$ and $\Delta y = 0,2$

$$\frac{\partial z}{\partial x} = 1 - \frac{1}{2\sqrt{x^2 + y^2}} \cdot 2x = 1 - \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial z}{\partial y} = 1 - \frac{y}{\sqrt{x^2 + y^2}}$$

$$dz = \left(1 - \frac{x}{\sqrt{x^2 + y^2}}\right)\Delta x + \left(1 - \frac{y}{\sqrt{x^2 + y^2}}\right)\Delta y$$

$$dz = \left(1 - \frac{3}{\sqrt{3^2 + 4^2}}\right) \cdot 0,1 + \left(1 - \frac{4}{\sqrt{3^2 + 4^2}}\right) \cdot 0,2 = 0,08$$

23. Evaluate dz and Δz if $z = \dfrac{xy}{x + y}$, $x = 1$, $y = 2$, $\Delta x = 0,4$ and $\Delta y = 0,2$

$$f(x+\Delta x, y+\Delta y) \approx f(x,y) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

Using the total differential compute the approximate value

24. $1,04^{2,02}$

$$f(x,y) = x^y$$

$$x = 1 \qquad \Delta x = 0,04 \qquad y = 2 \qquad \Delta y = 0,02$$

$$f(1,2) = 1^2 = 1$$

$$\frac{\partial f}{\partial x} = yx^{y-1}\Big|_{(1;2)} = 2 \cdot 1^1 = 2$$

$$\frac{\partial f}{\partial y} = x^y \ln x\Big|_{(1;2)} = 1^2 \cdot \ln 1 = 0$$

$$1,04^{2,02} \approx 1 + 2 \cdot 0,04 + 0 \cdot 0,02 = 1,08$$

25. $\arctan \frac{1,02}{0,96}$ and $\arctan \frac{0,97}{1,04}$

$$\arctan \frac{1,02}{0,96}$$

$$f(x,y) = \arctan \frac{x}{y}$$

$$x = 1 \qquad \Delta x = 0,02 \qquad y = 1 \qquad \Delta y = -0,04$$

$$f(1;1) = \arctan \frac{1}{1} = \frac{\pi}{4}$$

$$\frac{\partial f}{\partial x} = \frac{1}{1+(\frac{x}{y})^2} \cdot \frac{1}{y} = \frac{y^2}{y^2+x^2} \cdot \frac{1}{y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial f}{\partial x}\Big|_{(1;1)} = \frac{1}{2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+(\frac{x}{y})^2} \cdot (-\frac{x}{y^2}) = -\frac{x}{x^2+y^2}$$

$$\frac{\partial f}{\partial y}\Big|_{(1;1)} = -\frac{1}{2}$$

$$\arctan\frac{1,02}{0,96} \approx \frac{\pi}{4} + \frac{1}{2}\cdot 0,02 - \frac{1}{2}\cdot(-0,04) = \frac{\pi}{4} + 0,03$$

$$\arctan\frac{0,97}{1,04}$$

$$x = 1 \qquad \Delta x = -0,03 \qquad y = 1 \qquad \Delta y = 0,04$$

$$\arctan\frac{0,97}{1,04} \approx \frac{\pi}{4} + \frac{1}{2}\cdot(-0,03) - \frac{1}{2}\cdot 0,04 = \frac{\pi}{4} - 0,035$$

26. $\ln(\sqrt[3]{1,03} + \sqrt[4]{0,98} - 1)$

$$f(x,y) = \ln(\sqrt[3]{x} + \sqrt[4]{y} - 1)$$

$$x = 1 \qquad \Delta x = 0,03 \qquad y = 1 \qquad \Delta y = -0,02$$

$$f(1;1) = \ln(\sqrt[3]{1} + \sqrt[4]{1} - 1) = \ln 1 = 0$$

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt[3]{x} + \sqrt[4]{y} - 1}\cdot\frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}(\sqrt[3]{x} + \sqrt[4]{y} - 1)}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1;1)} = \frac{1}{3}$$

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt[3]{x} + \sqrt[4]{y} - 1}\cdot\frac{1}{4}y^{-\frac{3}{4}} = \frac{1}{4\sqrt[4]{y^3}(\sqrt[3]{x} + \sqrt[4]{y} - 1)}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1;1)} = \frac{1}{4}$$

$$\ln(\sqrt[3]{1,03} + \sqrt[4]{0,98} - 1) \approx 0 + \frac{1}{3}\cdot 0,03 + \frac{1}{4}\cdot(-0,02) = 0,005$$

Derivative and partial derivatives of implicit functions

$$F(x,y)=0$$

$$\frac{dy}{dx}=-\frac{F'_x}{F'_y}$$

Find $\dfrac{dy}{dx}$ for

27. $xy-\ln y=a$

$$xy-\ln y-a=0$$

$$\frac{dy}{dx}=-\frac{y}{x-\frac{1}{y}}=-\frac{y^2}{xy-1}=\frac{y^2}{1-xy}$$

28. $x^3y-xy^3=a^4$

$$x^3y-xy^3-a^4=0$$

$$\frac{dy}{dx}=-\frac{3x^2y-y^3}{x^3-3xy^2}=-\frac{y(3x^2-y^2)}{x(x^2-3y^2)}$$

29. $x^2y^2-x^4-y^4=a^4$

30. $xe^y+ye^x-e^{xy}=0$

$$\frac{dy}{dx}=-\frac{e^y+ye^x-ye^{xy}}{xe^y+e^x-xe^{xy}}$$

31. Evaluate $\dfrac{dy}{dx}$ at the point $(a;a)$ if $x^4y+xy^4-ax^2y^2=a^5$

$$F(x, y, z) = 0$$

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z} \qquad \frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z}$$

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

32. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{1}{a^2} \cdot 2x}{\frac{1}{c^2} \cdot 2z} = -\frac{c^2 x}{a^2 z}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{1}{b^2} \cdot 2y}{\frac{1}{c^2} \cdot 2z} = -\frac{c^2 y}{b^2 z}$$

33. $z^3 + 3xyz = a^3$

$$z^3 + 3xyz - a^3 = 0$$

$$\frac{\partial z}{\partial x} = -\frac{3yz}{3z^2 + 3xy} = -\frac{yz}{z^2 + xy}$$

$$\frac{\partial z}{\partial y} = -\frac{3xz}{3z^2 + 3xy} = -\frac{xz}{z^2 + xy}$$

34. $e^z - xyz = 1$

$$e^z - xyz - 1 = 0$$

$$\frac{\partial z}{\partial x} = -\frac{-yz}{e^z - xy} = \frac{yz}{e^z - xy}$$

$$\frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}$$

Partial derivatives of a higher order

$$z = f(x, y) \qquad \frac{\partial z}{\partial x} \qquad \frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \qquad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial^3 z}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right)$$

$$\frac{\partial^3 z}{\partial y \partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y \partial x}\right)$$

35. Let $z = \dfrac{x-y}{x+y}$. Find $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial x \partial y}$, $\dfrac{\partial^2 z}{\partial y \partial x}$ and $\dfrac{\partial^2 z}{\partial y^2}$

36. Let $z = x^y$. Prove that $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$

$$\frac{\partial z}{\partial x} = yx^{y-1} \qquad \frac{\partial z}{\partial y} = x^y \ln x$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(yx^{y-1}\right) = x^{y-1} + yx^{y-1}\ln x = x^{y-1}(1 + y\ln x)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial x}\left(x^y \ln x\right) = yx^{y-1}\ln x + x^y \cdot \frac{1}{x} = x^{y-1}(y\ln x + 1)$$

37. Find $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial x \partial y}$ and $\dfrac{\partial^2 z}{\partial y^2}$ if $z = y^{\ln x}$

$$\frac{\partial z}{\partial x} = y^{\ln x}\ln y \cdot \frac{1}{x} \qquad \frac{\partial z}{\partial y} = \ln x \cdot y^{\ln x - 1}$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \ln y \cdot \frac{\partial}{\partial x}\left(y^{\ln x} \cdot \frac{1}{x}\right) = \\
&= \ln y \left(y^{\ln x}\ln y \cdot \frac{1}{x} \cdot \frac{1}{x} - y^{\ln x} \cdot \frac{1}{x^2}\right) = \\
&= \frac{1}{x^2} y^{\ln x}\ln y(\ln y - 1)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{1}{x} \cdot \frac{\partial}{\partial y}\left(y^{\ln x}\ln y\right) = \\
&= \frac{1}{x}\left(\ln x y^{\ln x - 1}\ln y + y^{\ln x} \cdot \frac{1}{y}\right) = \\
&= \frac{1}{xy} y^{\ln x}(\ln x \ln y + 1)
\end{aligned}$$

$$\frac{\partial^2 z}{\partial y^2} = \ln x \cdot \frac{\partial}{\partial y}\left(y^{\ln x - 1}\right) = \ln x(\ln x - 1)y^{\ln x - 2}$$

38. Find $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial x \partial y}$ and $\dfrac{\partial^2 z}{\partial y^2}$ if $z = \sin^2(ax + by)$

Here we use the formula

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\frac{\partial z}{\partial x} = 2\sin(ax + by)\cos(ax + by) \cdot a = a\sin(2ax + 2by)$$

$$\frac{\partial z}{\partial y} = 2\sin(ax + by)\cos(ax + by) \cdot b = b\sin(2ax + 2by)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(a\sin(2ax+2by)) = a\cos(2ax+2by)\cdot 2a = 2a^2\cos(2ax+2by)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}(a\sin(2ax+2by)) = a\cos(2ax+2by)\cdot 2b = 2ab\cos(2ax+2by)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(b\sin(2ax+2by)) = b\cos(2ax+2by)\cdot 2b = 2b^2\cos(2ax+2by)$$

39. Let $z = \dfrac{y}{y^2 - a^2x^2}$. Prove that $\dfrac{\partial^2 z}{\partial x^2} = a^2 \dfrac{\partial^2 z}{\partial y^2}$

40. Let $w = \ln \dfrac{1}{\sqrt{x^2 + y^2}}$. Prove that $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = 0$

41. Find $\dfrac{\partial^3 z}{\partial x^2 \partial y}$ for $z = e^{xy^2}$

$$\frac{\partial z}{\partial x} = e^{xy^2} \cdot y^2$$

$$\frac{\partial^2 z}{\partial x^2} = y^2 \cdot \frac{\partial}{\partial x}\left(e^{xy^2}\right) = y^2 e^{xy^2} \cdot y^2 = y^4 e^{xy^2}$$

$$\begin{aligned}\frac{\partial^3 z}{\partial x^2 \partial y} &= \frac{\partial}{\partial y}\left(y^4 e^{xy^2}\right) = 4y^3 e^{xy^2} + y^4 e^{xy^2} \cdot 2xy = \\ &= 2y^3 e^{xy^2}(2 + xy^2)\end{aligned}$$

42. Find $\dfrac{\partial^3 z}{\partial x \partial y^2}$ for $z = \sin(xy)$

43. Find $\dfrac{\partial^3 w}{\partial x \partial y \partial z}$ for $w = e^{xyz}$

$$\frac{\partial w}{\partial x} = e^{xyz} \cdot yz$$

$$\begin{aligned}\frac{\partial^2 w}{\partial x \partial y} &= \frac{\partial}{\partial y}\left(e^{xyz} \cdot yz\right) = \\ &= e^{xyz} \cdot xz \cdot yz + e^{xyz} \cdot z = e^{xyz} \cdot xyz^2 + e^{xyz} \cdot z\end{aligned}$$

$$\begin{aligned}\frac{\partial^3 w}{\partial x \partial y \partial z} &= \frac{\partial}{\partial z}\left(e^{xyz} \cdot xyz^2 + e^{xyz} \cdot z\right) = \\ &= e^{xyz} \cdot xy \cdot xyz^2 + e^{xyz} \cdot 2xyz + e^{xyz} \cdot xy \cdot z + e^{xyz} = \\ &= e^{xyz}(x^2y^2z^2 + 3xyz + 1)\end{aligned}$$

44. Let $r = \sqrt{x^2 + y^2 + z^2}$. Prove that

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}$$

and

$$\frac{\partial^2 (\ln r)}{\partial x^2} + \frac{\partial^2 (\ln r)}{\partial y^2} + \frac{\partial^2 (\ln r)}{\partial z^2} = \frac{1}{r^2}$$