Application of partial derivatives

Directional derivative

To find the directional derivative of $z = f(x, y)$ at the point P in direction of vector $\vec{s} = (\Delta x; \Delta y)$ we use the formula

$$\frac{\partial z}{\partial \vec{s}} = \frac{\partial z}{\partial x} \cos\alpha + \frac{\partial z}{\partial y} \cos\beta$$

where partial derivatives are evaluated at P and directional cosines $\cos\alpha$ and $\cos\beta$ are coordinates of unit vector $\overrightarrow{s^0}$ in direction of vector $\vec{s}$, i.e.

$$\overrightarrow{s^0} = \left(\frac{\Delta x}{\Delta \vec{s}}; \frac{\Delta y}{\Delta \vec{s}}\right) = (\cos\alpha; \cos\beta)$$

The directional derivative of function of three variables $w = f(x, y, z)$ will be evaluated by the formula

$$\frac{\partial w}{\partial \vec{s}} = \frac{\partial w}{\partial x} \cos\alpha + \frac{\partial w}{\partial y} \cos\beta + \frac{\partial w}{\partial z} \cos\gamma$$

1. Find the derivative $z = x^3 - 3x^2y + 3xy^2 + 1$ at the point $M(3; 1)$ towards the point $N(6; 5)$

Solution

First we evaluate partial derivatives at M

$$\frac{\partial z}{\partial x} = 3x^2 - 6xy + 3y^2\Big|_{M(3;1)} = 12$$

and

$$\frac{\partial z}{\partial y} = -3x^2 + 6xy\Big|_{M(3;1)} = -9$$

Next, the length on vector $\vec{s} = \overrightarrow{MN} = (3; 4)$ is $\Delta\vec{s} = 5$. Consequently directionl cosines are

$$(\cos\alpha; \cos\beta) = \left(\frac{3}{5}; \frac{4}{5}\right)$$

and by the formula

$$\frac{\partial z}{\partial \vec{s}} = 12 \cdot \frac{3}{5} - 9 \cdot \frac{4}{5} = 0$$

2. Find the derivative $z = \ln(e^x + e^y)$ at the origin in the direction that forms the angle $30°$ with x-axis.

Solution

$$\frac{\partial z}{\partial x} = \frac{e^x}{e^x + e^y}\bigg|_{O(0;0)} = \frac{1}{2}$$

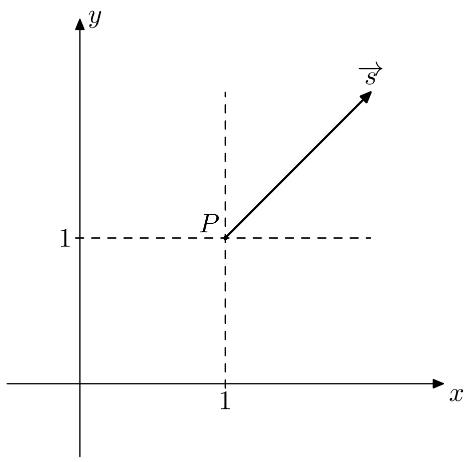
$$\frac{\partial z}{\partial y} = \frac{e^y}{e^x + e^y}\bigg|_{O(0;0)} = \frac{1}{2}$$

$$\overrightarrow{s^0} = (\cos 30°; \cos 60°) = \left(\frac{\sqrt{3}}{2}; \frac{1}{2}\right)$$

$$\frac{\partial z}{\partial \vec{s}} = \frac{\sqrt{3}+1}{4}$$

3. Find the derivative $z = \arctan(xy)$ at point $P(1;1)$ in direction of the bisectrix of the first quarter of coordinate plane.

Solution



First we evaluate partial derivatives at P

$$\frac{\partial z}{\partial x} = \frac{y}{1 + x^2y^2}\bigg|_{P} = \frac{1}{2}$$

$$\frac{\partial z}{\partial y} = \frac{x}{1 + x^2y^2}\bigg|_{P} = \frac{1}{2}$$

The angle between bisectrix of the first quarter of coordinate plane and x-axis (y-axis) is $45°$, so

$$\overrightarrow{s^0} = (\cos 45°; \cos 45°) = \left(\frac{\sqrt{2}}{2}; \frac{\sqrt{2}}{2}\right)$$

and by the formula

$$\frac{\partial z}{\partial \vec{s}} = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$$

4. Find the derivative $w = xyz$ at point $A(5; 1; 2)$ in direction that leads from A to $B(9; 4; 14)$.

Solution

Partial derivatives of w at A are

$$\frac{\partial w}{\partial x} = yz\Big|_A = 2$$

$$\frac{\partial w}{\partial y} = xz\Big|_A = 10$$

$$\frac{\partial w}{\partial z} = xy\Big|_A = 5$$

The length of direction vector $\vec{s} = \overrightarrow{AB} = (4; 3; 12)$ is 13, i.e. directional cosines are

$$(\cos \alpha; \cos \beta; \cos \gamma) = \left(\frac{4}{13}; \frac{3}{13}; \frac{12}{13}\right)$$

and according to the formula for three-dimensional case

$$\frac{\partial z}{\partial \vec{s}} = 2 \cdot \frac{4}{13} + 10 \cdot \frac{3}{13} + 5 \cdot \frac{12}{13} = \frac{98}{13}$$

5. Find the derivative $w = \sin(yz) + \ln x^2$ at point $(1;1;\pi)$ in the direction of vector $\overrightarrow{s} = (1;1;-1)$.

Solution

$$\frac{\partial w}{\partial x} = \frac{2}{x}\bigg|_{(1;1;\pi)} = 2$$

$$\frac{\partial w}{\partial y} = z\cos(yz)\bigg|_{(1;1;\pi)} = -\pi$$

$$\frac{\partial w}{\partial z} = y\cos(yz)\bigg|_{(1;1;\pi)} = -1$$

$$\Delta\vec{s} = \sqrt{1+1+1} = \sqrt{3}$$

$$\overrightarrow{s^0} = \left(\frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}; -\frac{1}{\sqrt{3}}\right)$$

$$\frac{\partial w}{\partial \vec{s}} = 2\cdot\frac{1}{\sqrt{3}} - \pi\cdot\frac{1}{\sqrt{3}} + 1\cdot\frac{1}{\sqrt{3}} = \frac{3-\pi}{\sqrt{3}}$$

6. Find the derivative $w = xy^2 + z^3 - xyz$ at point $(2;1;1)$ in the direction that forms the angles $60°$, $45°$ and $60°$ with x-, y- and z-axes, respectively.

Solution

$$\frac{\partial w}{\partial x} = y^2 - yz\bigg|_{(2;1;1)} = 0$$

$$\frac{\partial w}{\partial y} = 2xy - xz\bigg|_{(2;1;1)} = 2$$

$$\frac{\partial w}{\partial z} = 3z^2 - xy\bigg|_{(2;1;1)} = 1$$

$$\overrightarrow{s^0} = \left(\frac{1}{2}; \frac{\sqrt{2}}{2}; \frac{1}{2}\right)$$

$$\frac{\partial w}{\partial \vec{s}} = \sqrt{2} + \frac{1}{2}$$

Gradient of scalar field

The gradient of scalar field $z = f(x, y)$ is

$$\operatorname{grad} z = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$$

The gradient of scalar field $w = f(x, y, z)$ is

$$\operatorname{grad} w = \left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\right)$$

7. Find the gradient of scalar field $z = \sqrt{4 + x^2 + y^2}$ at the point $A(2; 1)$

Solution We find the partial derivatives

$$\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{4 + x^2 + y^2}} \cdot 2x = \frac{x}{\sqrt{4 + x^2 + y^2}}$$

and

$$\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{4 + x^2 + y^2}} \cdot 2y = \frac{y}{\sqrt{4 + x^2 + y^2}}$$

Now, grad z at $A(2; 1)$ is

$$\operatorname{grad} z = \left(\frac{x}{\sqrt{4 + x^2 + y^2}}; \frac{y}{\sqrt{4 + x^2 + y^2}}\right)\Bigg|_A = \left(\frac{2}{3}; \frac{1}{3}\right)$$

8. Find the gradient of scalar field $z = \arcsin \dfrac{x}{x + y}$ at the point $B(1; 1)$

9. Find grad z for $z = \arctan \dfrac{y}{x}$

Solution

$$\operatorname{grad} z = \left(-\frac{y}{x^2 + y^2}; \frac{x}{x^2 + y^2}\right)$$

10. Find grad u for $u = \sqrt{x^2 + y^2 + z^2}$

Solution

$$\operatorname{grad} z = \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}; \frac{y}{\sqrt{x^2 + y^2 + z^2}}; \frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x; y; z)$$

For exercises 5., 6. and 7. let's recall a conclusion: *the directional derivative has the greatest value in the direction of the gradient and equals to the length of the gradient.*

11. Find the greatest ascent on the surface $z = x^y$ at the point $(2; 2; 4)$

Solution. Let us find the gradient at $(2; 2)$

$$\operatorname{grad} z = (yx^{y-1}; x^y \ln x)\Big|_{(2;2)} = (4; 4\ln 2)$$

and the length of this vector

$$|\operatorname{grad} z| = 4\sqrt{1 + \ln^2 2}$$

This length is the greatest ascent on the surface.

12. Find the greatest rate of growth of $z = \ln(x^2 + 4y^2)$ at the point $(6; 4; \ln 100)$

Solution. The gradient

$$\operatorname{grad} z = \left(\frac{2x}{x^2 + 4y^2}; \frac{8y}{x^2 + 4y^2}\right)\Big|_{(6;4)} = \left(\frac{3}{25}; \frac{8}{25}\right) = \frac{1}{25}(3; 8)$$

and the length of gradient

$$|\operatorname{grad} z| = \frac{\sqrt{73}}{25}$$

is the greatest rate of growth at $(6; 4; \ln 100)$.

13. Find the greatest rate of change of $w = x \sin z - y \cos z$ at the point $O(0; 0; 0)$

Solution. The gradient

$$\operatorname{grad} w = (\sin z; -\cos z; x\cos z + y \sin z)\Big|_{O} = (0; -1; 0)$$

and the length of gradient $|\operatorname{grad} w| = 1$ is the greatest rate of change at the origin.

Divergence and curl of vector field

The divergence of a vector field

$$\overrightarrow{F} = (X(x,y,z); Y(x,y,z); Z(x,y,z))$$

is the scalar

$$\operatorname{div} \overrightarrow{F} = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}$$

and the curl of vector field $\overrightarrow{F}$ is the vector

$$\operatorname{curl} \overrightarrow{F} = \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}; \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}; \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right)$$

14. Find the divergence and curl of vector field $\overrightarrow{F} = (x^2yz; xy^2z; xyz^2)$

Solution.

In this exercise $X = x^2yz$, $Y = xy^2z$ and $Z = xyz^2$. Thus, the divergence

$$\operatorname{div} \overrightarrow{F} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xyz^2) = 2xyz + 2xyz + 2xyz = 6xyz$$

and the curl

$$\begin{aligned} \operatorname{curl} \overrightarrow{F} &= \left(\frac{\partial}{\partial y}(xyz^2) - \frac{\partial}{\partial z}(xy^2z); \frac{\partial}{\partial z}(x^2yz) - \frac{\partial}{\partial x}(xyz^2); \frac{\partial}{\partial x}(xy^2z) - \frac{\partial}{\partial y}(x^2yz) \right) = \\ &= (xz^2 - xy^2; x^2y - yz^2; y^2z - x^2z) = (x(z^2 - y^2); y(x^2 - z^2); z(y^2 - x^2)) \end{aligned}$$

15. Find the divergence and curl of vector field $\overrightarrow{F} = (x(y+z); y(x+z); z(x+y))$

Answer.

$$\operatorname{div} \overrightarrow{F} = 2(x+y+z)$$

$$\operatorname{curl} \overrightarrow{F} = (z-y; x-z; y-x)$$

16. Find $\operatorname{div} \operatorname{grad} w$ and $\operatorname{curl} \operatorname{grad} w$ for scalar field $w = \ln(x^2+y^2+z^2)$

Solution.

$$\operatorname{grad} w = \left(\frac{2x}{x^2+y^2+z^2}; \frac{2y}{x^2+y^2+z^2}; \frac{2z}{x^2+y^2+z^2} \right)$$

$$\frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2+z^2} \right) = \frac{2(x^2+y^2+z^2) - 2x \cdot 2x}{(x^2+y^2+z^2)^2} = \frac{2(y^2+z^2-x^2)}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2+z^2}\right)=\frac{2(x^2+y^2+z^2)-2y\cdot 2y}{(x^2+y^2+z^2)^2}=\frac{2(x^2+z^2-y^2)}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial}{\partial z}\left(\frac{2z}{x^2+y^2+z^2}\right)=\frac{2(x^2+y^2+z^2)-2z\cdot 2z}{(x^2+y^2+z^2)^2}=\frac{2(x^2+y^2-z^2)}{(x^2+y^2+z^2)^2}$$

$$\operatorname{div}\operatorname{grad} w=\frac{2(y^2+z^2-x^2+x^2+z^2-y^2+x^2+y^2-z^2)}{(x^2+y^2+z^2)^2}$$

$$\operatorname{div}\operatorname{grad} w=\frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2}=\frac{2}{x^2+y^2+z^2}$$

First coordinate of curl vector

$$\frac{\partial}{\partial y}\left(\frac{2z}{x^2+y^2+z^2}\right)-\frac{\partial}{\partial z}\left(\frac{2y}{x^2+y^2+z^2}\right)=$$
$$2z\cdot\left(-\frac{1}{(x^2+y^2+z^2)^2}\right)\cdot 2y-2y\left(-\frac{1}{(x^2+y^2+z^2)^2}\right)\cdot 2z=0$$

$$\operatorname{curl}\operatorname{grad} w=(0;0;0)$$

Gradient field is irrotational.

17. Find $\operatorname{div}(w\overrightarrow{F})$ if $w=\varphi(x,y,z)$ is scalar field $\overrightarrow{F}=(X,Y,Z)$ is vector field.

Solution.

$$\begin{aligned}\operatorname{div}(w\overrightarrow{F}) &= \operatorname{div}(wX;wY;wZ)=\frac{\partial}{\partial x}(wX)+\frac{\partial}{\partial y}(wY)+\frac{\partial}{\partial z}(wZ)=\\ &= \frac{\partial w}{\partial x}\cdot X+w\frac{\partial X}{\partial x}+\frac{\partial w}{\partial y}\cdot Y+w\frac{\partial Y}{\partial y}+\frac{\partial w}{\partial x}\cdot Z+w\frac{\partial Z}{\partial z}=\\ &= \frac{\partial w}{\partial x}\cdot X+\frac{\partial w}{\partial y}\cdot Y+\frac{\partial w}{\partial x}\cdot Z+w\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)=\\ &= \operatorname{grad} w\cdot\overrightarrow{F}+w\cdot\operatorname{div}\overrightarrow{F}\end{aligned}$$

18. Find $\operatorname{curl}(\overrightarrow{F}\times\overrightarrow{c})$ if $\overrightarrow{F}=(X,Y,Z)$ and $\overrightarrow{c}=(c_1;c_2;c_3)$ is a constant vector.

19. Prove that $\operatorname{curl}(w\overrightarrow{F}) = \operatorname{grad} w \times \overrightarrow{F} + w \cdot \operatorname{curl} \overrightarrow{F}$

20. Prove that $\operatorname{curl}\operatorname{curl} \overrightarrow{F} = \operatorname{grad}\operatorname{div} \overrightarrow{F} - \Delta\overrightarrow{F}$

Local extrema of function of two variables

Let $P_0 = (x_0; y_0)$ be a stationary point of function $z = f(x, y)$, i.e. a solution of the system of equations

$$\begin{cases} \dfrac{\partial z}{\partial x} = 0 \\[2ex] \dfrac{\partial z}{\partial y} = 0 \end{cases}$$

Let us denote the values of second order partial derivatives at P_0

$$A = \left.\frac{\partial^2 z}{\partial x^2}\right|_{P_0}, \quad B = \left.\frac{\partial^2 z}{\partial x \partial y}\right|_{P_0} \quad \text{and} \quad C = \left.\frac{\partial^2 z}{\partial y^2}\right|_{P_0}$$

Sufficient conditions for existence of a local extremum.

1. If $AC - B^2 > 0$ and $A < 0$ then the function $z = f(x, y)$ has a local maximum at P_0.
2. If $AC - B^2 > 0$ and $A > 0$ then the function $z = f(x, y)$ has a local minimum at P_0.
3. If $AC - B^2 < 0$ then the function $z = f(x, y)$ has no local extremum at P_0. The point P_0 is called the saddle point of function $z = f(x, y)$.

21. Find local extrema of function $z = 4(x - y) - x^2 - y^2$

Solution. Partial derivatives

$$\frac{\partial z}{\partial x} = 4 - 2x \qquad \frac{\partial z}{\partial y} = -4 - 2y$$

The system of equations

$$\begin{cases} 4 - 2x = 0 \\ -4 - 2y = 0 \end{cases}$$

has one solution

$$\begin{cases} x = 2 \\ y = -2 \end{cases}$$

i.e. the function has one stationary point $P_0(2;-2)$

$$\frac{\partial^2 z}{\partial x^2} = -2, \quad \frac{\partial^2 z}{\partial x \partial y} = 0 \text{ and } \frac{\partial^2 z}{\partial y^2} = -2$$

$$A = -2, \quad B = 0 \text{ and } C = -2$$

To apply the theorem, we evaluate

$$AC - B^2 = -2 \cdot (-2) - 0^2 = 4$$

Hence $AC - B^2 > 0$ and $A < 0$ and by the first statement of theorem the function has a local maximum at $(2;-2)$

Answer. The function has at $(2;-2)$ a local maximum $z_{max} = 8$

22. Find local extrema of function $z = x^2 + xy + y^2 + x - y + 1$

Solution. Partial derivatives

$$\frac{\partial z}{\partial x} = 2x + y + 1$$

$$\frac{\partial z}{\partial y} = x + 2y - 1$$

The system of equations

$$\begin{cases} 2x + y + 1 = 0 \\ x + 2y - 1 = 0 \end{cases}$$

has one solution $x = -1$ and $y = 1$, i.e. there is one stationary point $P(-1;1)$. Second order partial derivatives

$$\frac{\partial^2 z}{\partial x^2} = 2, \quad \frac{\partial^2 z}{\partial x \partial y} = 1 \text{ and } \frac{\partial^2 z}{\partial y^2} = 2$$

Because all of these three are constants (does not depend on point), we have

$$A = 2, \quad B = 1 \text{ and } C = 2$$

To apply the theorem, we evaluate

$$AC - B^2 = 2 \cdot 2 - 1^2 = 3$$

Hence $AC - B^2 > 0$ and $A > 0$ and by the second statement of theorem the function has a local minimum at $(-1; 1)$ and this local minimum equals

$$z_{min} = 0$$

23. Find local extrema of function $z = x^3 + y^2 - 6xy - 39x + 18y + 20$

Solution. Partial derivatives

$$\frac{\partial z}{\partial x} = 3x^2 - 6y - 39$$

$$\frac{\partial z}{\partial y} = 2y - 6x + 18$$

To find stationary points, we need to solve the system of equations

$$\begin{cases} 3x^2 - 6y - 39 = 0 \\ 2y - 6x + 18 = 0 \end{cases}$$

which is equivalet to

$$\begin{cases} x^2 - 2y - 13 = 0 \\ y - 3x + 9 = 0 \end{cases}$$

We solve the second equation for y, $y = 3x - 9$ and substitute y into the first equation. The result is a quadratic equation

$$x^2 - 2(3x - 9) - 13 = 0$$

or

$$x^2 - 6x + 5 = 0$$

which has two roots $x_1 = 1$ and $x_2 = 5$. Related values of y are $y_1 = -6$ and $y_2 = 6$. Hence, this function has two stationary points $P_1(1; -6)$ and $P_2(5; 6)$

Next, second order partial derivatives

$$\frac{\partial^2 z}{\partial x^2} = 6x, \quad \frac{\partial^2 z}{\partial x \partial y} = -6 \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = 2$$

We have two constants constants

$$B = -6 \quad \text{and} \quad C = 2$$

For the first point $P_1(1;-6)$ the value of A will be $A = 6 \cdot 1 = 6$ and

$$AC - B^2 = 6 \cdot 2 - (-6)^2 = -24$$

According to the third statement of theorem the function has no local extremum at $P_1(1;-6)$ or this point is a saddle point of function given.

For the second stationary point $P_2(5;6)$ the value of A will be $A = 6 \cdot 5 = 30$ and

$$AC - B^2 = 30 \cdot 2 - (-6)^2 = 24$$

According to the second statement of theorem the function has a local minimum at $P_2(5;6)$ and this local minimum equals

$$z_{min} = -86$$

24. Find local extrema of function $z = x^3 + 3xy^2 - 15x - 12y$

Solution. Partial derivatives

$$\frac{\partial z}{\partial x} = 3x^2 + 3y^2 - 15$$

$$\frac{\partial z}{\partial y} = 6xy - 12$$

To find stationary points, we have to to solve the system of equations

$$\begin{cases} 3x^2 + 3y^2 - 15 = 0 \\ 6xy - 12 = 0 \end{cases}$$

which is equivalet to

$$\begin{cases} x^2 + y^2 - 5 = 0 \\ xy - 2 = 0 \end{cases}$$

To use substitution, we solve the second equation for y, $y = \dfrac{2}{x}$ and substitute y into the first equation. The result is the equation

$$x^2 + \frac{4}{x^2} - 5 = 0$$

or

$$x^4 - 5x^2 + 4 = 0$$

This is a quadratic equation with respect to x^2 and has two roots $x^2 = 1$ and $x^2 = 4$. So we have four different roots of the equation $x_1 = 1$, $x_2 = -1$, $x_3 = 2$ and $x_4 = -2$. Related values of y are $y_1 = 2$ and

$y_2 = -2$, $y_3 = 1$ and $y_4 = -1$. Hence, this function has four stationary points $P_1(1;2)$, $P_2(-1;-2)$, $P_3(2;1)$ and $P_4(-2;-1)$

Second order partial derivatives

$$\frac{\partial^2 z}{\partial x^2} = 6x, \quad \frac{\partial^2 z}{\partial x \partial y} = 6y \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = 6x$$

For the first stationary point $P_1(1;2)$ we get $A = 6 \cdot 1 = 6$, $B = 6 \cdot 2 = 12$ and $C = 6 \cdot 1 = 6$. To apply the theorem we evaluate

$$AC - B^2 = 6 \cdot 6 - 12^2 = -108$$

and by theorem $P_1(1;2)$ is a saddle point of the function.

For the second stationary point $P_1(-1;-2)$ we get $A = 6 \cdot (-1) = -6$, $B = 6 \cdot (-2) = -12$ and $C = 6 \cdot (-1) = -6$. We evaluate again

$$AC - B^2 = -6 \cdot (-6) - (-12)^2 = -108$$

to conclude that $P_2(-1;-2)$ is another saddle point of the function.

For the third stationary point $P_3(2;1)$ we get $A = 6 \cdot 2 = 12$, $B = 6 \cdot 1 = 6$ and $C = 6 \cdot 2 = 12$. Let us evaluate

$$AC - B^2 = 12 \cdot 12 - 6^2 = 108$$

So, $AC - B^2 > 0$ and $A > 0$ and the function has at $P_3(2;1)$ a local minimum $z_{min} = -28$

For the fourth stationary point $P_4(-2;-1)$ we get $A = 6 \cdot (-2) = -12$, $B = 6 \cdot (-1) = -6$ and $C = 6 \cdot (-2) = -12$. In this case

$$AC - B^2 = (-12) \cdot (-12) - (-6)^2 = 108$$

So, $AC - B^2 > 0$ and $A < 0$ and the function has at $P_4(-2;-1)$ a local maximum $z_{max} = 28$

25. Find local extrema of function $z = \ln x + \ln y + \dfrac{2}{x^2} + \dfrac{8}{y^2}$

26. Prove that the function $z = x^2 + xy + y^2 + \dfrac{a^3}{x} + \dfrac{a^3}{y}$ has at the point $\left(\dfrac{a}{\sqrt[3]{3}}; \dfrac{a}{\sqrt[3]{3}}\right)$ local minimum.