2 Partial derivatives

2.1 Partial derivatives

Fix in the domain of the function of two variables $z = f(x, y)$ one point $P(x, y)$. Holding y constant and increasing the variable x by Δx we have the increment of the function $f(x, y)$

$$\Delta_x z = f(x + \Delta x, y) - f(x, y)$$

.

Definition 1. If there exists the limit

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \tag{2.1}$$

then this limit is called the *partial derivative* of the function $f(x, y)$ with respect to the variable x at the point (x, y).

The partial derivative with respect to x is denoted also z'_x, $f'_x(x, y)$, $\dfrac{\partial f}{\partial x}$.

Holding x constant and increasing the variable y by Δy we have the increment of the function $f(x, y)$ as $\Delta_y z = f(x, y + \Delta y) - f(x, y)$.

Definition 2. If there exists the limit

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \tag{2.2}$$

then this limit is called $f(x, y)$ the partial derivative of the function $f(x, y)$ with respect to the variable y at the point (x, y).

The possible alternate notations for partial derivatives with respect to y are z'_y, $f'_y(x, y)$, $\dfrac{\partial f}{\partial y}$.

If we find the partial derivative with respect to the variable x the variable y is treated as constant. The only variable in Definition 1 is Δx. As well, finding the partial derivative with respect to the variable y the variable x is treated as constant. The only variable in Definition 2 is Δy. We need to pay very close attention to which variable we are differentiating with respect to. This is important because we are going to treat the other variable as constant and then proceed with the derivative as if it was a function of a single variable. Consequently, all the rules of differentiation of functions of one variable hold if we find the partial derivatives.

Example 1. Find the partial derivatives with respect to both variables for the function $z = x^3 y - x^2 y^2$.

Finding the partial derivative with respect to x, y is treated as constant. Thus, by the difference rule an constant rule we obtain

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^3 y) - \frac{\partial}{\partial x}(x^2 y^2) = y\frac{\partial}{\partial x}(x^3) - y^2\frac{\partial}{\partial x}(x^2) = y \cdot 3x^2 - y^2 \cdot 2x = 3x^2 y - 2xy^2.$$

Finding the partial derivative with respect to y, x is treated as constant. By the rules of differentiation

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^3 y) - \frac{\partial}{\partial y}(x^2 y^2) = x^3\frac{\partial}{\partial y}(y) - x^2\frac{\partial}{\partial y}(y^2) = x^3 - x^2 \cdot 2y = x^3 - 2x^2 y$$

The chain rule is also still valid.

Example 2. Find the partial derivatives with respect to both variables for the function $z = \arctan\frac{x}{y}$.

The partial derivative with respect to x is

$$\frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{\partial z}{\partial x}\left(\frac{x}{y}\right) = \frac{y^2}{y^2+x^2} \cdot \frac{1}{y}\frac{\partial}{\partial x}(x) = \frac{y}{x^2+y^2}$$

The partial derivative with respect to y is

$$\begin{aligned} \frac{\partial z}{\partial y} &= \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{\partial z}{\partial y}\left(\frac{x}{y}\right) = \frac{y^2}{y^2+x^2} \cdot x\frac{\partial}{\partial y}\left(\frac{1}{y}\right) \\ &= \frac{y^2}{x^2+y^2} \cdot \left(-\frac{x}{y^2}\right) = -\frac{x}{x^2+y^2} \end{aligned}$$

The partial derivatives of the function of three variables $w = f(x, y, z)$ with respect to variables x, y and z are defined as the limits

$$\frac{\partial w}{\partial x} = \lim_{\Delta x \to 0}\frac{\Delta_x w}{\Delta x} = \lim_{\Delta x \to 0}\frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$\frac{\partial w}{\partial y} = \lim_{\Delta y \to 0}\frac{\Delta_y w}{\Delta y} = \lim_{\Delta y \to 0}\frac{f(x, y+\Delta y, z) - f(x, y, z)}{\Delta y}$$

and

$$\frac{\partial w}{\partial z} = \lim_{\Delta z \to 0}\frac{\Delta_z w}{\Delta z} = \lim_{\Delta z \to 0}\frac{f(x, y, z+\Delta z) - f(x, y, z)}{\Delta z}$$

If we find the partial derivative with respect to one independent variable, the other independent variables are treated as constants.

Example 3. Find the partial derivatives with respect to all three independent variables for the function $w = x^{y^z}$.

Finding the partial derivative with respect to x, we have the power function with constant exponent y^z, therefore,

$$\frac{\partial w}{\partial x} = y^z x^{y^z - 1}$$

To find the partial derivative with respect to y we use the chain rule. The outside function is the exponential function with constant base x and the variable exponent y^z, which is the power function with respect to y. By the chain rule

$$\frac{\partial w}{\partial y} = x^{y^z} \ln x \cdot z y^{z-1}$$

To find the partial derivative with respect to z we use the chain rule again. The outside function is the exponential function with constant base x. The inside function is another exponential function y^z with the constant base y. Thus

$$\frac{\partial w}{\partial z} = x^{y^z} \ln x \cdot y^z \ln y$$

2.2 Total increment and total differential

Let us fix one point $P(x, y)$ in the domain of function $z = f(x, y)$. Assume that the function $f(x, y)$ is continuous and has the continuous partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at the point $P(x, y)$ and in some neighborhood of this point.

It is possible to prove that total increment

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

can be represented as

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \tag{2.3}$$

where ε_1 and ε_2 are two infinitesimals as $(\Delta x; \Delta y) \to (0; 0)$ i.e.

$$\lim_{(\Delta x;\Delta y)\to(0;0)} \varepsilon_1 = \lim_{(\Delta x;\Delta y)\to(0;0)} \varepsilon_2 = 0$$

In subsection 1.4 we have used the notation $\Delta\rho = \sqrt{\Delta x^2 + \Delta y^2}$. The conditions

$$\left|\frac{\Delta x}{\Delta \rho}\right| \le 1$$

and

$$\left|\frac{\Delta y}{\Delta\rho}\right| \le 1$$

mean that these are the bounded quantities. Thus, $\varepsilon_1 \dfrac{\Delta x}{\Delta\rho}$ and $\varepsilon_2 \dfrac{\Delta y}{\Delta\rho}$ are infinitesimals as the products of the infinitesimals and a bounded quantities. Thus, the limit

$$\lim_{\Delta\rho\to 0} \frac{\varepsilon_1 \Delta x + \varepsilon_2 \Delta y}{\Delta\rho} = \lim_{\Delta\rho\to 0} \varepsilon_1 \frac{\Delta x}{\Delta\rho} + \lim_{\Delta\rho\to 0} \varepsilon_2 \frac{\Delta y}{\Delta\rho} = 0$$

which means that $\varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ is an infinitesimal of the higher order with respect to $\Delta\rho$, i.e. with respect to Δx and Δy.

After that in the representation (2.3) $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are the values of partial derivatives at the fixed point P i.e. the real numbers. Hence, the first sum

$$\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \tag{2.4}$$

is linear with respect to Δx and Δy.

Definition. The linear part (2.4) of the total increment (2.3) is called the *total differential* of the function $z = f(x, y)$ and denoted by dz.

According to the definition

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

For the function $z = x$ the partial derivatives $\dfrac{\partial z}{\partial x} = 1$, $\dfrac{\partial z}{\partial y} = 0$ and $dz = dx = \Delta x$.

For the function $z = y$ the partial derivatives $\dfrac{\partial z}{\partial x} = 0$, $\dfrac{\partial z}{\partial y} = 1$ and $dz = dy = \Delta y$.

Consequently for the independent variables x and y the notions of differential and increment coincide and the total differential can be re-written as

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy. \tag{2.5}$$

Example 1. Find the total differential for the function $z = \arctan \dfrac{x}{y}$. Using the partial derivatives found in Example 2 of subsection 1.5, we obtain

$$dz = \frac{y}{x^2 + y^2}dx - \frac{x}{x^2 + y^2}dy = \frac{ydx - xdy}{x^2 + y^2}$$

Example 2. Evaluate the total increment and the total differential for the function $z = \sqrt{x^2 + y^2}$, if $x = 3$, $y = 4$, $\Delta x = 0,2$ and $\Delta y = 0,1$.

By the formula of the total increment of the function we get

$$\Delta z = \sqrt{3,2^2 + 4,1^2} - \sqrt{3^2 + 4^4} = \sqrt{27,05} - \sqrt{25} = 0,20096$$

To evaluate the total differential we find

$$\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x^2 + y^2}} \cdot 2x = \frac{x}{\sqrt{x^2 + y^2}}$$

and

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

Then

$$dz = \frac{3}{\sqrt{3^2 + 4^2}} \cdot 0,2 + \frac{4}{\sqrt{3^2 + 4^2}} \cdot 0,1 = \frac{0,6}{5} + \frac{0,4}{5} = 0,2$$

We see that the difference between the total increment and the total differential is less than $0,001$, which is less by two orders of values with respect to Δx and Δy.

The last fact gives us the possibility to compute the approximate values of functions of two variables using the total differential. If Δx and Δy are sufficiently small, then Δz and dz differ by the quantity, which is the infinitesimal of a higher order with respect to Δx and Δy. We can write

$$\Delta z \approx dz$$

or

$$f(x + \Delta x, y + \Delta y) - f(x, y) \approx \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

This gives us the formula of approximate computation

$$f(x + \Delta x, y + \Delta y) \approx f(x, y) + \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y \qquad (2.6)$$

Example 3. Using the total differential, compute $2,03^3 \cdot 0,96^2$.

Here we choose the function $f(x, y) = x^3 y^2$ and the values $x = 2$, $y = 1$, $\Delta x = 0,03$ and $\Delta y = -0,04$. The partial derivatives are

$$\frac{\partial f}{\partial x} = 3x^2 y^2$$

and

$$\frac{\partial f}{\partial y} = 2x^3 y$$

The value of the function at the point chosen $f(2,1) = 8 \cdot 1 = 8$ and the values of partial derivatives are $\dfrac{\partial f}{\partial x} = 3 \cdot 4 \cdot 1 = 12$ and $\dfrac{\partial f}{\partial y} = 2 \cdot 8 \cdot 1 = 16$. By the formula (2.9)

$$(2 + 0,03)^3 \cdot (1 - 0,04)^2 = 8 + 12 \cdot 0,03 - 16 \cdot 0,04 = 7,72$$

Suppose that the function of three variables $w = f(x, y, z)$ and the partial derivatives $\dfrac{\partial w}{\partial x}$, $\dfrac{\partial w}{\partial y}$ and $\dfrac{\partial w}{\partial z}$ are continuous at the point $P(x, y, z)$ and in some neighborhood of this point. Analogously to the formula (2.3) it is possible to prove that the total increment of the function can be expressed as

$$\Delta w = \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z + \alpha\Delta x + \beta\Delta y + \gamma\Delta z, \qquad (2.7)$$

where $\alpha\Delta x + \beta\Delta y + \gamma\Delta z$ is an infinitesimal of a higher order with respect to $\Delta\rho = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$. The expression

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz \qquad (2.8)$$

is called the total differential of the function $w = f(x, y, z)$. Again, for the independent variables x, y and z the notions of the increment and differential coincide, i.e. $dx = \Delta x$, $dy = \Delta y$ and $dz = \Delta z$.

Example 4. Find the total differential for the function $w = x^{y^z}$.

Using the partial derivatives found in Example 3 of subsection 1.5, we obtain

$$\begin{aligned} dw &= y^z x^{y^z - 1} dx + x^{y^z} \ln x \cdot z y^{z-1} dy + x^{y^z} \ln x \cdot y^z \ln y = \\ &= y^z x^{y^z} \left(\frac{dx}{x} + \frac{z \ln x dy}{y} + \ln x \ln y \right) \end{aligned}$$

As well as for the function of two variables there holds the formula of approximate computation

$$f(x + \Delta x, y + \Delta y, z + \Delta z) \approx f(x, y, z) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z \qquad (2.9)$$

2.3 Partial derivatives of implicit function

Consider the function

$$F(x, y) = 0 \tag{2.10}$$

given implicitly. This equation determines the variable y as the function of x (in general case not one-valued).

Suppose that the function $F(x, y)$ is continuous and it has the continuous partial derivatives at the point $P(x, y)$ and in some neighborhood of this point. In addition suppose that at $P(x, y)$ the partial derivative $F'_y(x, y) \neq 0$. Let us deduce the formula to find the derivative $\frac{dy}{dx}$, using the partial derivatives of the function $F(x, y)$.

Let us fix the point $P(x, y)$ on the graph of given function. The coordinates of this point satisfy the equation (2.10). Change x by Δx and find on the graph the value of $y + \Delta y$ related to $x + \Delta x$. As $Q(x + \Delta x, y + \Delta y)$ is a point on the graph again, the coordinates of this point also satisfy the equation

$$F(x + \Delta x, y + \Delta y) = 0. \tag{2.11}$$

Subtracting from the equation (2.11) the equation (2.10), we obtain

$$F(x + \Delta x, y + \Delta y) - F(x, y) = 0$$

The left side of the last equality is the total increment of the function $F(x, y)$ and the equality can be re-written

$$\Delta F = 0$$

Because of the assumptions made in the beginning of this subsection this equality converts by (2.3) to

$$\frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial y}\Delta y + \alpha\Delta x + \beta\Delta y = 0$$

which yields

$$\left(\frac{\partial F}{\partial y} + \beta\right)\Delta y = -\left(\frac{\partial F}{\partial x} + \alpha\right)\Delta x$$

or

$$\frac{\Delta y}{\Delta x} = -\frac{\dfrac{\partial F}{\partial x} + \alpha}{\dfrac{\partial F}{\partial y} + \beta}$$

Find the limits of both sides of this equality as $\Delta x \to 0$. The limit of the left side is by the definition of the derivative $\dfrac{dy}{dx}$. The function is continuous, consequently if $\Delta x \to 0$ then $\Delta y \to 0$. Knowing that α and β are the infinitesimals as $(\Delta x, \Delta y) \to (0; 0)$, that is $\lim\limits_{\Delta x \to 0} \alpha = 0$ and $\lim\limits_{\Delta x \to 0} \beta = 0$, the limit of the right side of the equality is

$$-\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}}$$

Thus, to find the derivative of the function given implicitly we have the formula

$$\frac{dy}{dx} = -\frac{F'_x}{F'_y} \tag{2.12}$$

Example 1. Find $\dfrac{dy}{dx}$ for $x^4 + y^4 - a^2x^2y^2 = 0$.

Here $F(x, y) = x^4 + y^4 - a^2x^2y^2$, so $F'_x = 4x^3 - 2a^2xy^2$ and $F'_y = 4y^3 - 2a^2x^2y$. By the formula (2.12)

$$\frac{dy}{dx} = -\frac{4x^3 - 2a^2xy^2}{4y^3 - 2a^2x^2y} = -\frac{x(2x^2 - a^2y^2)}{y(2y^2 - a^2x^2)}.$$

The equation $F(x, y, z) = 0$ relates to pairs of (x, y) some value(s) of the variable z. In other words, this equation defines z as a function of x and y. Assume that the function $F(x, y, z)$ is continuous and has the continuous partial derivatives $\dfrac{\partial F}{\partial x}$, $\dfrac{\partial F}{\partial y}$ and $\dfrac{\partial F}{\partial z}$ at the point $P(x, y, z)$ and in some neighborhood of this point. Moreover assume that $F'_z(x, y, z) \neq 0$ at $P(x, y, z)$.

If we find the partial derivative of the function z with respect to x the variable y is treated as constant. In this case in the equation $F(x, y, z) = 0$ there are only two variables x and z and by (2.12) we obtain

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z} \tag{2.13}$$

If we repeat this reasoning for y we have

$$\frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z} \tag{2.14}$$

Example 2. Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the function of two variables $x^2 + y^2 + z^2 = r^2$ given implicitly.

As $F'_x = 2x$, $F'_y = 2y$ and $F'_z = 2z$ we obtain by the formula (2.13) the partial derivative

$$\frac{\partial z}{\partial x} = -\frac{x}{z}$$

and by the formula (2.14) the partial derivative

$$\frac{\partial z}{\partial y} = -\frac{y}{z}$$

2.4 Partial derivatives of composite functions

Suppose that the variable z is a function of two variables u and v, denote $z = f(u, v)$, and u and v are the functions of two independent variables x and y, denote $u = \varphi(x, y)$ and $v = \psi(x, y)$. Then z is a composite function with respect to x and y, i.e.

$$z = f(\varphi(x, y), \psi(x, y)) = F(x, y)$$

Let us fix a point $P(x, y)$ in the common domain of the functions $u = \varphi(x, y)$ and $v = \psi(x, y)$. Then the related point (u, v) in the (u, v)-plane is also fixed. Suppose that the functions u and v are continuous and have the continuous partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ at the point $P(x, y)$ and in some neighborhood of this point. Also assume that the function z is continuous and has the continuous partial derivatives $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at the related point (u, v) and in some neighborhood of this point.

The partial derivative of the composite function $z = F(x, y)$ with respect to x will be found by the formula

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \tag{2.15}$$

The partial derivative of the composite function z with respect to the variable y will be found by

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \tag{2.16}$$

Example 1. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for $z = \ln(u^2+v)$, $u = e^{x+y^2}$ and $v = x^2+y$.

According to the formulas (2.15) and (2.16) we have to find six partial derivatives

$$\frac{\partial z}{\partial u} = \frac{2u}{u^2+v}, \quad \frac{\partial z}{\partial v} = \frac{1}{u^2+v};$$
$$\frac{\partial u}{\partial x} = e^{x+y^2}, \quad \frac{\partial u}{\partial y} = 2ye^{x+y^2};$$
$$\frac{\partial v}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 1;$$

By (2.15) we have

$$\frac{\partial z}{\partial x} = \frac{2u}{u^2+v}e^{x+y^2} + \frac{1}{u^2+v}2x = \frac{2}{u^2+v}(ue^{x+y^2}+x)$$

and by (2.16)

$$\frac{\partial z}{\partial y} = \frac{2u}{u^2+v}2ye^{x+y^2} + \frac{1}{u^2+v} = \frac{1}{u^2+v}(4uye^{x+y^2}+1)$$

Remark. If z is a function of three variables $z = f(u, v, w)$ and in addition to the u and v there is $w = \chi(x, y)$, then the partial derivatives of the composite function z with respect to the variables x and y can be found by the formulas

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial x} \tag{2.17}$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial y} \tag{2.18}$$

Next, let z be a function of three variables x, u and v $z = f(x, u, v)$, where $u = \varphi(x)$ and $v = \psi(x)$. In this case z is a composite function of one variable x

$$z = f(x, \varphi(x), \psi(x))$$

The derivative of that function $\dfrac{dz}{dx}$ we obtain using (2.17). As the derivative $\dfrac{dx}{dx} = 1$ and u and v are the functions of one variable, then

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial u}\frac{du}{dx} + \frac{\partial z}{\partial v}\frac{dv}{dx}. \tag{2.19}$$

The derivative in (2.19) is called the *total derivative*.

Example 2. Find $\frac{dz}{dx}$ for $z = x^2 + \sqrt{y}$ and $y = x^2 + 1$.

Here z is the function of two variables x and y, where y is the function of the variable x. In this case the formula (2.19) gives

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} = 2x + \frac{1}{2\sqrt{y}} \cdot 2x = x\left(2 + \frac{1}{\sqrt{y}}\right) = x\left(2 + \frac{1}{\sqrt{x^2+1}}\right).$$

2.5 Higher order partial derivatives

As we have seen in many examples, the partial derivatives of the function $z = f(x, y)$ $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are in general functions of two variables again. Thus, it is possible to differentiate both of them with respect to x and y.

Definition 1. The partial derivative with respect to x of the partial derivative $\frac{\partial z}{\partial x}$ is called the *second order partial derivative with respect to x* and denoted $\frac{\partial^2 z}{\partial x^2}$ (to be read *de-squared-zed de-ex-squared*), that means

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

Definition 2. The partial derivative with respect to y of the partial derivative $\frac{\partial z}{\partial x}$ is called the *second order partial derivative with respect to x and y* and denoted $\frac{\partial^2 z}{\partial x \partial y}$ (to be read *de-squared-zed de-ex-de-y*). By this definition

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

Definition 3. The partial derivative with respect to x of the partial derivative $\frac{\partial z}{\partial y}$ is called the *second order partial derivative with respect to y and x* and denoted $\frac{\partial^2 z}{\partial y \partial x}$, that is

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

Definition 4. The partial derivative with respect to y of the partial derivative $\frac{\partial z}{\partial y}$ is called the *second order partial derivative with respect to y* and denoted $\dfrac{\partial^2 z}{\partial y^2}$, i.e

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

The second and third second order partial derivatives are often called *mixed partial derivatives* since we are taking derivatives with respect to more than one variable.

The second order partial derivatives are denoted also z''_{xx}, z''_{xy}, z''_{yx} and z''_{yy} or $f''_{xx}(x, y)$, $f''_{xy}(x, y)$, $f''_{yx}(x, y)$ and $f''_{yy}(x, y)$.

The second order partial derivatives are the functions of two variables x and y again. Hence, all four second order partial derivatives can be differentiated with respect to x and y. So we define eight third order partial derivatives

$$\frac{\partial^3 z}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right), \quad \frac{\partial^3 z}{\partial x^2 \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x^2}\right)$$

$$\frac{\partial^3 z}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x \partial y}\right), \quad \frac{\partial^3 z}{\partial x \partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x \partial y}\right)$$

$$\frac{\partial^3 z}{\partial y \partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y \partial x}\right), \quad \frac{\partial^3 z}{\partial y \partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y \partial x}\right)$$

$$\frac{\partial^3 z}{\partial y^2 \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right), \quad \frac{\partial^3 z}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y^2}\right)$$

Example 1. Find all second order partial derivatives for $z = \arctan\dfrac{x}{y}$.

In Example 2 of subsection 6.5 we have found

$$\frac{\partial z}{\partial x} = \frac{y}{x^2 + y^2} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{x}{x^2 + y^2}$$

We find

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{y}{x^2 + y^2}\right) = y\frac{\partial}{\partial x}\left(\frac{1}{x^2 + y^2}\right) = y\left(-\frac{2x}{(x^2 + y^2)^2}\right) = -\frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right) = \frac{x^2 + y^2 - y \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial x}\left(-\frac{x}{x^2 + y^2}\right) = -\frac{x^2 + y^2 - x \cdot 2x}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{x}{x^2 + y^2}\right) = -x\frac{\partial}{\partial y}\left(\frac{1}{x^2 + y^2}\right) = -x\left(-\frac{2y}{(x^2 + y^2)^2}\right) = \frac{2xy}{(x^2 + y^2)^2}$$

These results suggest a question, are the mixed second order partial derivatives

$$\frac{\partial^2 z}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 z}{\partial y \partial x}$$

equal. The next theorem says that if the function is smooth enough this will always be the case.

Theorem. If the function $z = f(x, y)$ and its partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ are continuous at the point P and on some neighborhood of this point, then at the point P

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

This theorem says that if the partial derivatives to be evaluated are continuous, then the result of repeated differentiation is independent of the order in which it is performed.

Therefore, if the partial derivatives involved are continuous, the also holds

$$\frac{\partial^4 z}{\partial x \partial y \partial x \partial y} = \frac{\partial^4 z}{\partial x^2 \partial y^2} = \frac{\partial^4 z}{\partial y^2 \partial x^2}$$

Analogous theorem is valid also for the functions of three etc. variables.

Example 2. Find the third order partial derivatives $\frac{\partial^3 w}{\partial x \partial y \partial z}$ and $\frac{\partial^3 w}{\partial z \partial x \partial y}$ for the function of three variables $w = e^x \sin(yz)$.

First we find

$$\frac{\partial w}{\partial x} = e^x \sin(yz)$$

second

$$\frac{\partial^2 w}{\partial x \partial y} = e^x \cos(yz) \cdot z = z e^x \cos(yz)$$

and third

$$\frac{\partial^3 w}{\partial x \partial y \partial z} = e^x \cos(yz) + z(-e^x \sin(yz)) \cdot y = e^x [\cos(yz) - yz \sin(yz)]$$

To find the second third order partial derivative, we find

$$\frac{\partial w}{\partial z} = y e^x \cos(yz)$$

next

$$\frac{\partial^2 w}{\partial z \partial x} = y e^x \cos(yz)$$

and finally

$$\frac{\partial^3 w}{\partial z \partial x \partial y} = e^x \cos(yz) - ye^x \sin(yz) \cdot z = e^x[\cos(yz) - yz\sin(yz)]$$

2.6 Directional derivative

Up to now for the function of two variables $z = f(x, y)$ we've only looked at the two partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Recall that these derivatives represent the rate of change of f as we vary x (holding y fixed) and as we vary y (holding x fixed) respectively. We now need to discuss how to find the rate of change of $f(x, y)$ if we allow both x and y to change simultaneously. In other words how to find the rate of change of $f(x, y)$ in the direction of vector $\overrightarrow{s} = (\Delta x, \Delta y)$.

The goal is to obtain the formula to compute the derivative of the function $z = f(x, y)$ at the point $P(x, y)$ in the direction of the vector $\overrightarrow{s} = (\Delta x, \Delta y)$.

Assume that the function $z = f(x, y)$ and its partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are continuous at P and in some neighborhood of this point.

Denote the length of the vector $\overrightarrow{s}$ by $\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$. By the (2.3) the total increment of the function has the form

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and ε_2 are infinitesimals as $\Delta s \to 0$. Dividing the last equality by the length of the vector $\overrightarrow{s}$ gives

$$\frac{\Delta z}{\Delta s} = \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta s} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta s} + \varepsilon_1 \frac{\Delta x}{\Delta s} + \varepsilon_2 \frac{\Delta y}{\Delta s}$$

The ratios $\frac{\Delta x}{\Delta s}$ and $\frac{\Delta y}{\Delta s}$ are the coordinates of the unit vector $\overrightarrow{s^0}$ in direction of the vector $\overrightarrow{s}$. Denoting by α and β the angles that $\overrightarrow{s}$ forms with the coordinate axes, it's obvious that

$$\frac{\Delta x}{\Delta s} = \cos\alpha \quad \text{and} \quad \frac{\Delta y}{\Delta s} = \cos\beta$$

Therefore, these ratios, i.e. the coordinates of the unit vector in direction of the vector $\overrightarrow{s}$ are called the *directional cosines* of that vector.

Definition. The limit

$$\lim_{\Delta s \to 0} \frac{\Delta z}{\Delta s}$$

is called the derivative of z at the point P in the direction of the vector $\overrightarrow{s}$ and denoted $\dfrac{\partial z}{\partial \overrightarrow{s}}$. Since

$$\lim_{\Delta s \to 0} \left(\varepsilon_1 \frac{\Delta x}{\Delta s} + \varepsilon_2 \frac{\Delta y}{\Delta s} \right) = 0$$

we have the formula to compute the directional derivative

$$\frac{\partial z}{\partial \overrightarrow{s}} = \frac{\partial z}{\partial x} \cos \alpha + \frac{\partial z}{\partial y} \cos \beta \tag{2.20}$$

Example 1. Find the derivatives of the function $z = x^2 + y^2$ at the point $P(1;1)$ in directions of vectors $\overrightarrow{s_1} = (1;1)$ and $\overrightarrow{s_2} = (1;-1)$.

First we evaluate the partial derivatives of z at P

$$\frac{\partial z}{\partial x} = 2x\bigg|_P = 2$$

and

$$\frac{\partial z}{\partial y} = 2y\bigg|_P = 2$$

The length of the vector $\overrightarrow{s_1}$ is $\Delta s_1 = \sqrt{2}$, the directional cosines are $\cos \alpha = \dfrac{1}{\sqrt{2}}$ and $\cos \beta = \dfrac{1}{\sqrt{2}}$. Hence,

$$\frac{\partial z}{\partial \overrightarrow{s_1}} = 2 \cdot \frac{1}{\sqrt{2}} + 2 \cdot \frac{1}{\sqrt{2}} = 2\sqrt{2}$$

The length of the vector $\overrightarrow{s_2}$ is $\Delta s_2 = \sqrt{2}$, the directional cosines are $\cos \alpha = \dfrac{1}{\sqrt{2}}$ and $\cos \beta = -\dfrac{1}{\sqrt{2}}$. Thus,

$$\frac{\partial z}{\partial \overrightarrow{s_2}} = 2 \cdot \frac{1}{\sqrt{2}} - 2 \cdot \frac{1}{\sqrt{2}} = 0$$

Starting from the same point in the xy plane and moving in different directions, we get the different results. Thus, the directional derivative has no meaning without specifying the direction. The directional derivative gives us the instantaneous rate of change of the given function of two variables at a certain point in the pre-scribed direction.

Partial derivatives with respect to x and y are special cases of the directional derivative. If the given vector $\overrightarrow{s}$ points in direction of x-axis then $\alpha = 0$, $\beta = \dfrac{\pi}{2}$, $\cos\alpha = 1$ and $\cos\beta = 0$. Hence,

$$\frac{\partial z}{\partial \overrightarrow{s}} = \frac{\partial z}{\partial x}$$

If the given vector $\overrightarrow{s}$ points in direction of y-axis then $\alpha = \dfrac{\pi}{2}$, $\beta = 0$, $\cos\alpha = 0$ and $\cos\beta = 1$. It follows

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial y}$$

Thus, the directional derivative in the direction of x axis is the partial derivative with respect to x and the directional derivative in the direction of y-axis is the partial derivative with respect to y.

The directional derivative of the function of three variables $w = f(x, y, z)$ at the point $P(x, y, z)$ in the direction of the vector $\overrightarrow{s} = (\Delta x, \Delta y, \Delta z)$ can be found by the similar formula. Let α, β and γ denote the angles between the vector $\overrightarrow{s}$ and x-axis, y-axis and z-axis respectively. Then the directional cosines of the vector $\overrightarrow{s}$ are $\cos\alpha$, $\cos\beta$ and $\cos\gamma$. The directional derivative is computed by the formula

$$\frac{\partial w}{\partial \overrightarrow{s}} = \frac{\partial w}{\partial x}\cos\alpha + \frac{\partial w}{\partial y}\cos\beta + \frac{\partial w}{\partial z}\cos\gamma \tag{2.21}$$

Example 2. Find the directional derivative of the function $w = xy + xz + yz$ at the point $P(1; 1; 2)$ in the direction of the vector that makes with the coordinate axes the angles 60°, 60° and 45° respectively.

Find the partial derivatives at the point P

$$\frac{\partial w}{\partial x} = y + z\Big|_P = 3, \quad \frac{\partial w}{\partial y} = x + z\Big|_P = 3$$

and

$$\frac{\partial w}{\partial z} = x + y\Big|_P = 2$$

and the directional cosines

$$\overrightarrow{s^{\circ}} = (\cos 60^{\circ}; \cos 60^{\circ}; \cos 45^{\circ}) = \left(\frac{1}{2}; \frac{1}{2}; \frac{\sqrt{2}}{2}\right).$$

By the formula (2.21) we obtain

$$\frac{\partial w}{\partial \overrightarrow{s}} = 3 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} + 2 \cdot \frac{\sqrt{2}}{2} = 3 + \sqrt{2}$$